Lecture 11: Analysis of the Childress cell problem and stability of cellular flows

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1 Introduction

In the first part of this lecture we will discuss the Childress analysis of the cell flow problem and apply to it variational methods. Furtheremore the general case of coupled Childress cells will be briefly analyzed. In the second part of the lecture we will discuss the stability of 2D cell flows for the forced Navier–Stokes equation.

2 Childress analysis of of the advection-diffusion problem for a simple cell flow

In the previous lecture we have seen that the multiscale analysis gives us the following large scale equation

$$\nabla\left[(\epsilon \boldsymbol{I} + \Psi)(\nabla\chi + e)\right] = 0, \tag{1}$$

where $\boldsymbol{I}$ is the unit matrix and Ψ is the matrix given by

$$\Psi(x,y) = \begin{pmatrix} 0 & -\psi(x,y) \\ \psi(x,y) & 0. \end{pmatrix} \tag{2}$$

The effective diffusivity is equal to

$$\sigma^*_\epsilon(\boldsymbol{e}) = \langle(\epsilon\boldsymbol{I}+\Psi)(\nabla\chi+\boldsymbol{e})\cdot\boldsymbol{e}\rangle = \sigma^*_\epsilon(\boldsymbol{e}) = \langle(\epsilon\boldsymbol{I}+\Psi)(\nabla\chi+\boldsymbol{e})\cdot(\nabla\chi+\boldsymbol{e}\rangle = \epsilon+\epsilon\langle\nabla\chi\cdot\nabla\chi\rangle. \tag{3}$$

The background flow is assumed to be given by a very simple velocity field $\boldsymbol{u} = (-\partial_y\psi, \partial\psi)$ with stream function

$$\psi(x,y) = \sin x \sin y, \tag{4}$$

which is represented on Figure 1. Note that due to the symmetries of the flow (4) it is sufficient to consider a quarter of the original cell $[0,2\pi]\times[0,2\pi]$. Indeed, a fluid particle which is initially contained in the cell $[0,\pi]\times[0,\pi]$ will stay in this cell for all times, see Figure 1.

Equation (1) can be also written as

$$\epsilon\Delta\chi + \boldsymbol{u}\cdot\nabla\chi + \boldsymbol{e}\cdot\boldsymbol{u} = 0. \tag{5}$$

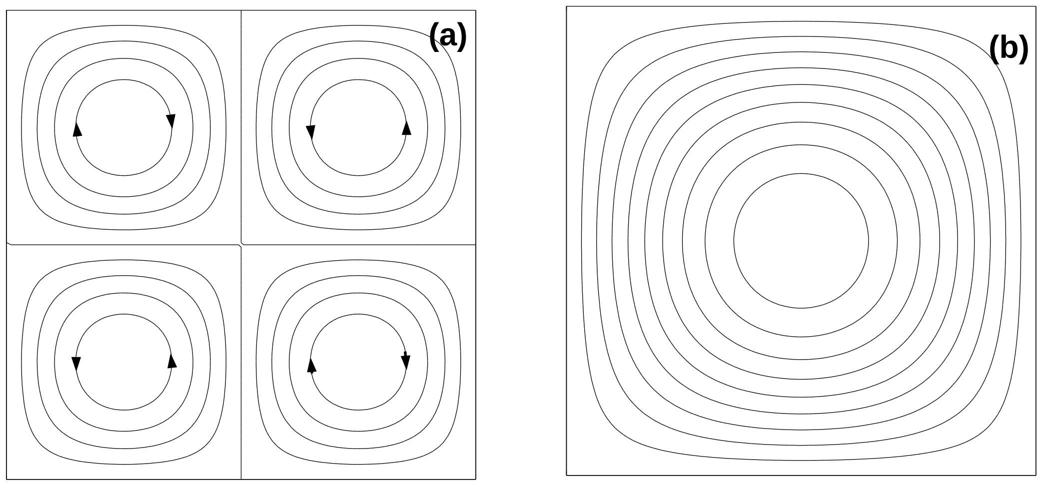

Figure 1: Simple cellular flow with stream function $\psi(x, y) = \sin x \sin y$ within the cell $[0, 2\pi] \times [0, 2\pi]$ (a) and the quarter cell $[0, \pi] \times [0, \pi]$ (b).

Chose $\boldsymbol{e} = (1, 0)$ and let $\rho = \chi + x$. Then we obtain the equation

$$\epsilon \Delta \rho + \boldsymbol{u} \cdot \nabla \rho = 0. \tag{6}$$

The boundary conditions are specified as follows

$$\rho(0, y) = 0 \qquad \rho(\pi, y) = 0; \tag{7}$$
$$\frac{\partial}{\partial y}\rho(x, 0) = 0 \qquad \frac{\partial}{\partial y}\rho(x, \pi) = 0. \tag{8}$$

The effective diffusivity can be calculated as follows

$$\sigma_\epsilon^* = \frac{\epsilon}{\pi^2} \int_0^\pi \int_0^\pi (\nabla \rho)^2 \, dx dy, \tag{9}$$

where $\sigma_\epsilon^*(\boldsymbol{e}_1) = \sigma_\epsilon^*$. To calculate the effective diffusivity (9) boundary layer theory can be applied [1]. On dimensional grounds the thickness of the boundary layer is expected to be of order $\sqrt{\epsilon}$. Indeed, the boundary layer can be estimated by equating the convection time scale $t_{\text{conv}} \sim L/U_0$ and and the diffusion time scale $t_{\text{diff}} \sim l^2/\nu$ (U_0 is the characteristic velocity and ν is the viscosity). The quantity l/L is the width of the boundary layer. Equating t_{conv} and t_{diff} we obtain $\frac{L}{U_0} \sim \frac{l^2}{\nu}$ and $\frac{L\nu}{U_0 L^2} \sim \left(\frac{l}{L}\right)^2$. Since $\epsilon = \frac{1}{Pe} \sim \left(\frac{l}{L}\right)^2$ it follows that the width of the boundary laye is given by $\frac{l}{L} \sim \sqrt{\epsilon}$. The same arguments apply to the case of more general periodic flows (discussed in [2]) such as the one given by the stream function

$$\psi(x, y) = \sin x \sin y + \delta \cos x \cos y, \tag{10}$$

see Figure 2.

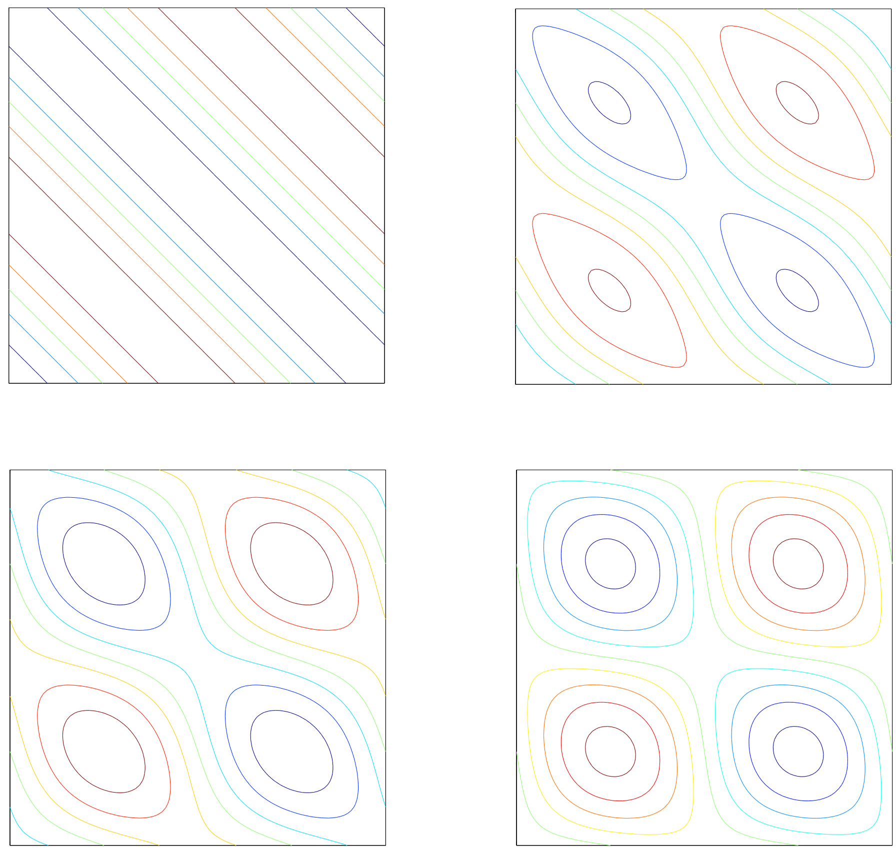

Figure 2: Cellular flows with stream function $\psi(x,y) = \sin x \sin y + \delta \cos x \cos y$ for different δ's.

Cellular flows connected with each other by slight random deformations of saddle points have been considered by Isichenko in [3]. The deformation is assumed to be of the form

$$\psi(x,y) = \sin x \sin y + \delta \tilde{\psi}(x,y), \tag{11}$$

where the parameter δ is assumed to be small and the function $\tilde{\psi}$ is random with certain properties [4].

In the periodic case the effictive diffusivity can be estimated by using the fact that ρ changes significantly only in the boundary layer. Therefore $\nabla\rho$ is of the order of $1/\sqrt{\epsilon}$ and $\sigma^*_\epsilon \sim \epsilon \left(\frac{1}{\sqrt{\epsilon}}\right)^2 \sqrt{\epsilon} \sim \sqrt{\epsilon}$.

More precise results can be obtained by using the boundary layer method. We introduce boundary coordinates

$$(x,y) \longrightarrow (\psi,\theta), \quad 0 \le \psi \le 1, \;\; -4 \le \theta \le 4. \tag{12}$$

Note that ψ is just the value of the stream function which is equal to zero on the boundary.

Furthermore , the level lines of ψ and θ are orthogonal

$$\nabla\psi \cdot \nabla\theta = 0, \quad |\nabla\psi| = |\nabla\theta| \text{ on } \psi = 0. \tag{13}$$

It is suitable to rescale the coordinate ψ in the neighborhood of the boundary. We define

$$(h, \theta) = \left(\frac{\psi}{\sqrt{\epsilon}}, \theta\right). \tag{14}$$

The standard chain rule yields

$$\frac{\partial\rho}{\partial x} = \frac{\partial h}{\partial x}\frac{\partial\rho}{\partial h} + \frac{\partial\theta}{\partial x}\frac{\partial\rho}{\partial\theta}$$

and

$$\rho_{xx} = h_{xx}\rho_x + h_x^2\rho_{hh} + 2h_x\theta_x\rho_{h,\theta} + \theta_{xx}\rho_\theta + \theta_x^2\rho_{\theta\theta}.$$

For $\epsilon \ll 1$ the left hand side of (6) can be written as

$$\epsilon(\rho_x x + \rho_y y) \to \frac{\epsilon}{\sqrt{\epsilon}}\Delta\psi\rho_h + |\nabla\psi|^2\rho_{hh} + \epsilon\Delta\theta\rho_\theta + \epsilon(\nabla\theta)^2\rho_{\theta\theta}.$$

Form the condition (13) follows

$$-\psi_y\rho_x + \psi_x\rho_y = -\nabla^\perp\psi \cdot \nabla\theta\rho_\theta = |\nabla\psi|^2\rho_\theta + \text{h.o.t.}$$

The boundary layer equation has the form

$$\rho_{hh} + \rho_\theta = 0, \quad h > 0, \ -4 \le \theta \le 4 \tag{15}$$

with boundary conditions

$$\rho(0,\theta) = 0, \quad \text{for } 0 \le \theta \le 2 \tag{16}$$

$$\rho(0,\theta) = \pi, \quad \text{for } -4 \le \theta \le -2 \tag{17}$$

$$\tfrac{\partial\rho}{\partial n} = 0, \quad \text{for } -2 \le \theta \le 0 \text{ and } 2 \le \theta 4. \tag{18}$$

Finally, we obtain the Childress equation

$$\frac{1}{\sqrt{\epsilon}}\sigma_\epsilon^* \to \frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 \rho_n^2 dh d\theta. \tag{19}$$

This problem has been treated by A. Soward in [5].

Finally, let us remark that boundary layer coordinates can be used to give an estimation of the scaling of σ_ϵ^* in the case of random flows. As before we suppose that the boundary is given by the level set $\psi = 0$. However, due to the randomness of the flow this boundary has a complicated fractal structure. For small ψ let the characteristic velocity at ψ be denoted as $U(\psi)$ and the width of the boundary layer by $l(\psi)$. Just as in the case of periodic flows we equate the diffusion time scale to the convection time scale $U^2(\psi)/\epsilon \sim l(\psi)/U_0$. Percolation methods can be applied to calculate the width of the boundary layer in dependence on ψ

[4]. This gives $l(\psi) \sim \psi^{-7/4}$. Since the velocity is assumed to be smooth it follows that $U(\psi) \sim \psi$ and $U_0 = O(1)$. Therefore

$$\psi \sim \epsilon^{4/15}. \tag{20}$$

Supposing that the gradient of ρ is of order $\frac{1}{\psi}$ we obtain from (9)

$$\sigma^*_\epsilon \sim \epsilon \left(\frac{1}{\psi}\right)^2 l(\psi)U(\psi) \sim \epsilon^{3/13}, \tag{21}$$

where $l(\psi)U(\psi)$ is the area of the boundary layer. Therefore, for random flows the effective diffusivity scales like $\epsilon^{\frac{3}{13}}$.

3 Variational analysis

The discussion of variational method results in this section is largely based on [2]. Denoting $E^+_{\boldsymbol{e}_1} = \nabla\chi + \boldsymbol{e}_1$ equation (1) becomes

$$\nabla \cdot (\boldsymbol{I} + \Psi)E^+_{\boldsymbol{e}_1} = 0 \tag{22}$$

and $E^+_{\boldsymbol{e}_1}$ satisfies conditions $\nabla \times E^+_{\boldsymbol{e}_1} = 0$ and $\langle E^+_{\boldsymbol{e}_1}\rangle = 0$. We also consider the adjoint problem

$$\nabla \cdot (\boldsymbol{I} - \Psi)E^-_{\boldsymbol{e}_2} = 0, \tag{23}$$

with $\nabla \times E^-_{\boldsymbol{e}_2} = 0$ and $\langle E^-_{\boldsymbol{e}_2}\rangle = 0$. For convenience of notation define

$$D^+_{\boldsymbol{e}_1} = (\boldsymbol{I} + \Psi)E^+_{\boldsymbol{e}_1}, \qquad E^-_{\boldsymbol{e}_2} = (\boldsymbol{I} - \Psi)E^-_{\boldsymbol{e}_2}. \tag{24}$$

Then the effective diffusivity becomes

$$\sigma^*(\boldsymbol{e}_1, \boldsymbol{e}_2) = \langle D^+_{\boldsymbol{e}_1} \cdot \boldsymbol{e}_2\rangle. \tag{25}$$

Define now

$$E'_{12} = \frac{1}{2}(E^+_{\boldsymbol{e}_1} - E^-_{\boldsymbol{e}_2}) \qquad D'_{12} = \frac{1}{2}(D^+_{\boldsymbol{e}_1} - D^-_{\boldsymbol{e}_2}), \tag{26}$$

$$E_{12} = \frac{1}{2}(E^+_{\boldsymbol{e}_1} + E^-_{\boldsymbol{e}_2}) \qquad D_{12} = \frac{1}{2}(D^+_{\boldsymbol{e}_1} + D^-_{\boldsymbol{e}_2}), \tag{27}$$

$$\tag{28}$$

It follows that

$$D'_{12} = E'_{12} + \Psi E_{12}, \qquad \nabla \cdot D'_{12} = 0, \;\; \nabla \times E'_{12} = 0 \tag{29}$$

$$D_{12} = E_{12} + \Psi E'_{12}, \qquad \nabla \cdot D_{12} = 0, \;\; \nabla \times E_{12} = 0. \tag{30}$$

The effective diffusivity can be written as

$$\sigma^* = \langle D^+_{\boldsymbol{e}_1} \cdot \boldsymbol{e}_2\rangle = \langle\frac{1}{2}\langle D^+_{\boldsymbol{e}_1} \cdot \boldsymbol{e}_2\rangle + \frac{1}{2}\langle D^-_{\boldsymbol{e}_2} \cdot \boldsymbol{e}_1\rangle = \frac{1}{2}\langle D^+_{\boldsymbol{e}_1} \cdot E^-_{\boldsymbol{e}_2}\rangle + \frac{1}{2}\langle D^-_{\boldsymbol{e}_2} \cdot E^+_{\boldsymbol{e}_1}\rangle =$$

$$\frac{1}{4}\langle (D^+_{\boldsymbol{e}_1} + D^-_{\boldsymbol{e}_2})(E^+_{\boldsymbol{e}_1} + E^-_{\boldsymbol{e}_2})\rangle - \frac{1}{4}\langle (D^+_{\boldsymbol{e}_1} - D^-_{\boldsymbol{e}_2})(E^+_{\boldsymbol{e}_1} - E^-_{\boldsymbol{e}_2})\rangle = \langle D_{12}\cdot E_{12}\rangle - \langle D'_{12}\cdot E'_{12}\rangle.$$

Then we obtain the following matrix equation

$$\sigma^* = \langle \begin{pmatrix} -\boldsymbol{I} & \Psi \\ \Psi & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} E'_{12} \\ E_{12} \end{pmatrix} \cdot \begin{pmatrix} E'_{12} \\ E_{12} \end{pmatrix} \rangle. \tag{31}$$

Note that the matrix $\begin{pmatrix} -\boldsymbol{I} & \Psi \\ \Psi & \boldsymbol{I} \end{pmatrix}$ is symmetric but indefinite.

Effective diffusivity can be computed as solution of the following variational problem

$$\sigma^{ast}(\boldsymbol{e}_1, \boldsymbol{e}_2) = \inf_{\langle F\rangle = \frac{\boldsymbol{e}_1+\boldsymbol{e}_2}{2},\, \nabla\times F=0} \ \sup_{\langle F'\rangle = \frac{\boldsymbol{e}_1-\boldsymbol{e}_2}{2},\, \nabla\times F'=0} \{\mathcal{A}(F,F')\} \tag{32}$$

where the matrix $\mathcal{A}(F,F')$ is given by

$$\mathcal{A}(F,F') = \langle \begin{pmatrix} -\boldsymbol{I} & \Psi \\ \Psi & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} F' \\ F \end{pmatrix} \cdot \begin{pmatrix} F' \\ F \end{pmatrix} \rangle.$$

The algebraic technique which underlies this calculation is that of a partial Legendre transform.

We will now give upper and lower bounds. First analyze the supremum. Consider the equation

$$\nabla F' + \nabla\cdot(\Psi F) = 0 \tag{33}$$

with

$$F' == \frac{\boldsymbol{e}_1 - \boldsymbol{e}_2}{2} - \Gamma\Psi F, \tag{34}$$

where $\Gamma\nabla\Delta^{-1}\nabla$ is the projection operator on the space of divergence-free vector fields. It is easily verified that (34) gives (33). Now we plug F' into the expression (32) setting $\boldsymbol{e}_1 = \boldsymbol{e}_2$. Then we obtain the following upper bound for the effective diffusivity

$$\sigma^*_\epsilon(\boldsymbol{e}) = \inf_{\nabla\times F=0,\, \langle F\rangle = \boldsymbol{e}} \{\epsilon\langle F\cdot F\rangle + \frac{1}{\epsilon}\langle \Gamma\Psi F\cdot\Gamma\Psi F\rangle. \tag{35}$$

Choose $F = \nabla f$ with $f = f(h,\theta)$. Then the first term $\langle F\cdot F\rangle$ in (35) gives

$$|\nabla f|^2 = |\nabla h|^2 \left(\frac{\partial f}{\partial h}\right)^2 + |\nabla\theta|^2 \left(\frac{\partial f}{\partial\theta}\right)^2, \tag{36}$$

where we have used (13). Since the second term in (36) is of order ϵ in comparison to the first term we obtain

$$\epsilon\langle F\cdot F\rangle \sim \frac{\epsilon}{\pi^2}\int_0^\infty \int_{-4}^4 |\nabla h|^2 \frac{1}{J(h,\theta)} \left(\frac{\partial f}{\partial h}\right)^2 dh\, d\theta \sim \frac{\sqrt{\epsilon}}{\pi^2}\int_0^\infty \int_{-4}^4 \left(\frac{\partial f}{\partial h}\right)^2 dh\, d\theta. \tag{37}$$

Here we have used the fact that near the boundary $J(h,\theta) \sim \sqrt{\epsilon}|\nabla h|^2$.

To calculate the second term in (35) suppose that $\frac{1}{\epsilon}\Gamma\Psi\nabla f = \nabla f'$ so that f' is the solution of the Poission equation

$$\epsilon\Delta f' = (-\psi_y, \psi_x)\cdot\nabla f. \tag{38}$$

The second term of (35) becomes now

$$\frac{1}{\epsilon}\langle \Gamma\Psi\nabla f \cdot \Gamma\Psi\nabla f\rangle = \epsilon\langle \nabla f' \cdot \nabla f'\rangle. \tag{39}$$

To obtain f' up to the leading order in ϵ it suffices to replace equation (38) by

$$\frac{\partial^2 f'}{\partial h^2} \sim \frac{\partial f'}{\partial \theta}, \tag{40}$$

where we have again used the fact that $J(h,\theta) \sim \sqrt{\epsilon}|\nabla h|^2$ near the boundary. Solving (40) by direct integration we can calculate the left hand side in (39) in the same way as we have done for $\epsilon\langle \nabla f \cdot \nabla f\rangle$. This gives

$$\frac{1}{\epsilon}\langle \Gamma\Psi\nabla f \cdot \Gamma\Psi\nabla f\rangle \sim \frac{\sqrt{\epsilon}}{\pi^2}\int_0^\infty \int_{-4}^4 \left(\int_\infty^h \frac{\partial f}{\partial h'} dh'\right)^2 dh\, d\theta. \tag{41}$$

Finally we obtain the following inequality

$$\lim_{\epsilon\to 0} \frac{1}{\sqrt{\epsilon}}\sigma_\epsilon^*(\boldsymbol{e}) \lesssim \frac{1}{\pi^2}\inf_f \int_0^\infty \int \left[\left(\frac{\partial f}{\partial h}\right)^2 + \left(\int_\infty^h \left(\frac{\partial f}{\partial \theta}\right) dh'\right)^2\right] dh d\theta. \tag{42}$$

In a similar a lower bound can be given. Note that Childress problem appears in both lower and upper bounds bounds and represents therefore an asymptotic relation [2].

4 Coupled Childress problems

In each cell we have different functions $f_i(h_i,\theta)$ and the following system of Childress equations

$$\frac{\partial^2 f_i}{\partial h^2} + \frac{\partial f_i}{\partial \theta} = 0, \quad h > 0,\ \ \theta \in [0, l_i]. \tag{43}$$

We have to impose the following boundary conditions: $f_i|_{\text{edges}} = f_{ik}(\theta)$, where k is one of the edges.

For common interior edges we have the conditions

$$\frac{\partial f_i}{\partial h} + \frac{\partial f_j}{\partial h}|_{h=0} = 0. \tag{44}$$

This allows us to construct a network approximation for convection-diffusion problems with many cells, see Figure 3.

5 The Stability of Cellular Flows

Let us consider the two dimensional Navier-Stokes equations driven by a spatially periodic force $F(y)$:

$$\begin{cases} \mathbf{U}_\tau + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla\tilde{p} + \frac{1}{Re}\Delta\mathbf{U} + F \\ \nabla\cdot\mathbf{U} = 0 \end{cases}$$

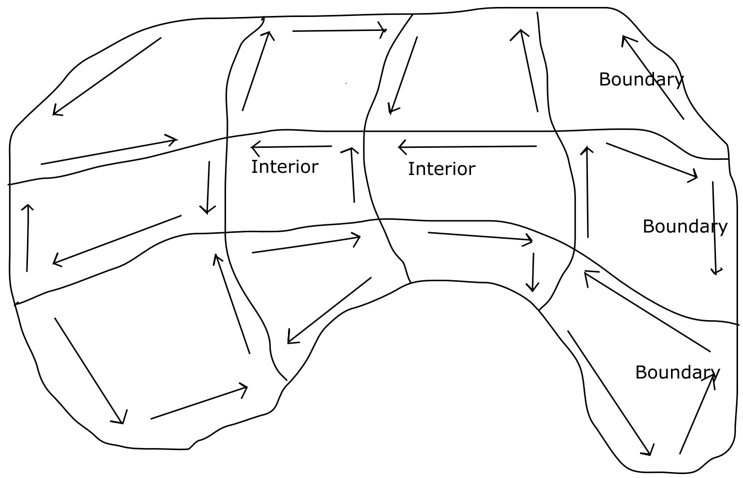

Figure 3: Network approximation for problem of many coupled cells.

where the Reynolds number $Re = \frac{UL}{\nu}$ is based on a length scale L which is proportional to the period of the forcing, taken to be equal to 2π.

As we are in two dimensions, the incompressibility condition $\nabla \cdot \mathbf{U} = 0$ implies that there exists a stream function Φ so that $\mathbf{U} = \binom{-\Phi_2}{\Phi_1}$.

Writing the Navier-Stokes equations above in terms of the stream function Φ we have:

$$\frac{\partial}{\partial t}\Delta\Phi + J_{yy}(\Phi, \Delta\Phi) = \frac{1}{Re}\Delta^2\Phi + f \tag{45}$$

where $J_{yy}(u, v) = -u_2 v_1 + u_1 v_2$. Here $f = -F_{1,2} + F_{2,1}$ is 2π periodic in R^2. It is chosen so that it gives rise to a stream function ϕ which is a time independent, mean-zero, periodic solution of the Navier-Stokes equations:

$$J_{yy}(\phi(y), \Delta\phi(y)) = \frac{1}{Re}\Delta^2\phi(y) + f(y) \tag{46}$$

Let $\Phi(\tau, y) = \phi(y) + \tilde{\Phi}(\tau, y)$ be a perturbation of the stationary solution $\phi(y)$. If the stream function of the basic flow is an *eddy* of size $k^{-\frac{1}{2}}$, that is if $\phi(y)$ is an eigenfunction of the Laplacian

$$\Delta\phi = -k\phi, \tag{47}$$

then the driving force $f(y)$ is

$$f(y) = -\frac{k^2}{Re}\phi(y)$$

and $\tilde{\Phi}(\tau, y)$ satisfies:

$$\partial_\tau \Delta\tilde{\Phi}(\tau, y) + J_{yy}(\phi(y), (k + \Delta)\tilde{\Phi}(\tau, y)) + J_{yy}(\tilde{\Phi}(\tau, y), \Delta\tilde{\Phi}(\tau, y)) = \frac{1}{Re}\Delta^2\tilde{\Phi}(\tau, y) \qquad (48)$$

What concerns us here is the stability of eddy flows like (46) and (47) subject to an initial *modulational* perturbation, a perturbation on a scale much larger than that of the eddy (see Dubrulle and Frisch([8] for references about previous works in this direction). For this purpose, we introduce a small parameter ϵ and define the large-scale time and space variable

$$t = \epsilon^2\tau, x = \epsilon y \qquad (49)$$

respectively and analyse a special class of asymptotic solutions of (48), where $\tilde{\Phi}(\tau, y) = \Psi^\epsilon(t, x)$ is expressed in the large-scale or slow variables as:

$$\Psi^\epsilon(t, x) = \Psi(t, x) + \epsilon\Psi^1(t, x, x/\epsilon) + \epsilon^2\Psi^2(t, x, x/\epsilon) + \dots \qquad (50)$$

One can derive (see [6] for details) from (48) the large scale modulational equation for $\Psi(t, x)$ in the vorticity form:

$$\partial_t\nabla^2\Psi(t, x) + \alpha^{nonlin}_{jikl}\nabla_{ji}(\nabla_k\Psi(t, x)\nabla_l\Psi(t, x)) = \nu_{jikl}\nabla_j\nabla_i\nabla_k\nabla_l\Psi(t, x) \qquad (51)$$

(where we used the convention $\nabla_i = \frac{\partial}{\partial x_i}$).

The coefficients ν_{jikl} are the tensor of eddy viscosity and α^{nonlin}_{jikl} are the effective coefficients of another tensor which we call the nonlinear α-tensor (see [6] for details). Both tensors are derived as necessary solvability conditions of auxiliary cell problems that guarantee the validity of the separation of scales for some finite time.

We will consider a family of cellular flows with a stream function

$$\phi = \sin(y_1)\sin(y_2) + \delta\cos(y_1)\cos(y_2), 0 \leq \delta \leq 1$$

All coefficients of the eddy viscosity tensor ν_{jikl} but one, called ν' can be computed analytically. The large-scale modulation equation corresponding to ν' is:

$$\frac{\partial}{\partial_t}\nabla^2\Psi + \frac{Re^2}{8}(\nabla_2^2 - \nabla_1^2)[\delta((\nabla_1\Psi)^2 + (\nabla_2\Psi)^2) + (1 + \delta^2)\nabla_1\Psi\nabla_2\Psi] + +J_{xx}(\Psi, \nabla^2\Psi)$$
$$= \frac{1}{Re}\nabla^4\Psi - \frac{Re}{8}[(\nabla_1 + \delta\nabla_2)^2 + (\delta\nabla_1 + \nabla_2)^2]\nabla^2\Psi + +(\frac{Re}{2}(1 + \delta^2) + \nu')(\nabla_2^2 - \nabla_1^2)\Psi \quad (52)$$

The ν' can be computed numerically for $Re \leq 32$, and for closed cellular flows $\phi = \sin(y_1)\sin(y_2)$ it can be shown that $\nu' = O(Re^{2.5})$ for large Re. This is done using an extension of the variational principles discussed earlier in this lecture (for details see [6]). Previously, Sivashinsky and Yakhot ([7]) and also Dubrulle and Frisch ([8]) have done a small Reynolds number linear stability analysis (see [7]), but in our case we are concerned with large Reynolds number flow.

The modulational perturbations of closed cellular flows ($\delta = 0$ in (5)) are much more stable than the shear cellular flows ($\delta = 1$ in (5)) for large Reynolds numbers. More specifically, exponential solutions $\Psi(t, x) = \exp(\sigma t)\exp(k_1x_1 + k_2x_2)$ are asymptotically unstable as $Re \to \infty$ *only* if $k_1 \approx \pm k_2$ for closed cellular flows. This result is to be contrasted with a similar stability result for shear flows, where exponential solutions are asymptotically unstable as $Re \to \infty$ if $C_1 \leq |k_1|/|k_2| \leq C_2$ where $C_1 = 1/C_2 \approx 0.45 \neq 1$. It can also be shown that because of the presence of $\nu' = O(Re^{2.5})$ for closed cellular flows, the stability at high Reynolds numbers is significantly better for flows with closed streamlines. Cell-like mesoscale ocean flows (which are at high Reynolds numers in the range of $10 - -10^3$) are close to closed celluar flows, and so the previous analysis may explain their persistance.

Notes by Ravi Srinivasan, Dani Zarnescu and Walter Pauls.

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