Lecture 12: Transcritical flow over an obstacle

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1 Introduction

The flow of a fluid over an obstacle is a classical and fundamental problem in fluid mechanics. When the flow velocity is near the local wave speed, obstacles in the path of this resonant flow tend to generate interesting wave fields around them. This lecture explores transcritical flow over a step primarily in the framework of the forced Korteweg-deVries equation using both asymptotic analysis and numerical simulations. This lecture follows from Lecture 11 in that resonant flow over a step may be resolved with undular bores.

2 Linearized shallow-water theory

Consider one-dimensional shallow-water flow past topography. The flow variables we are interested in are the total local depth $H = \zeta + h - F(x)$ and the depth-averaged horizontal velocity V. Here ζ is the surface elevation above the undisturbed depth h and the bottom is located at $z = -h + F(x)$ where $F(x)$ is the obstacle. The fully nonlinear shallow water equations for conservation of mass and momentum are

$$\zeta_t + (HV)_x = 0, \tag{1}$$

$$V_t + VV_x + g\zeta_x = -\frac{(H^2D^2H)_x}{3H} - \frac{(H^2D^2F)_x}{2H} - \frac{F_xD^2(\zeta + F)}{2}, \tag{2}$$

$$\text{where } D = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x}.$$

Equation (1) is exact, but equation (2) is a long-wave approximation; the terms on the right-hand side are the leading-order effects of wave dispersion. They form the Su-Gardner equations (also known as the Green-Naghdi equations).

If the Su-Gardner equations are linearized about the constant state U, h, where $V = U + u$, $|u| << V$, $|\zeta| << h$, they reduce to the forced linear wave equation

$$D_I^2\zeta - c^2\zeta_{xx} - U^2F_{xx},\ D_I = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x},\ c = \sqrt{gh}. \tag{3}$$

Here c is the linear long-wave speed, and a key parameter is the Froude number, $Fr = U/c$. Provided that background flow velocity is not critical ($Fr \neq 1$), as $t \to \infty$, there is a steady solution

$$\zeta = \frac{U^2}{U^2 - c^2}F(x). \tag{4}$$

The solution is a stationary depression over the obstacle for subcritical flow ($Fr < 1$), and a stationary elevation for supercritical flow ($Fr > 1$).

3 Forced Korteweg-deVries equation

The linear solution clearly fails as the flow nears criticality $Fr \approx 1$. The wave energy cannot propagate away from the obstacle. In this case, it is necessary to invoke weak nonlinearity and weak dispersion. This results in the forced Korteweg-deVries (fKdV) equation.

For water waves, the fKdV equation is, in non-dimensional form with scaling of characteristic length h and velocity c,

$$-\zeta_t - \Delta\zeta_x + \frac{3}{2}\zeta\zeta_x + \frac{1}{6}\zeta_{xxx} + \frac{1}{2}F_x = 0. \tag{5}$$

Here $\Delta = Fr - 1$ measures the degree of criticality, subcritical for $\Delta < 0$ or supercritical for $\Delta > 0$. The equation describes the usual KdV balance between nonlinearity, dispersion and time evolution, supplemented here by forcing and criticality. The asymptotic regime where equation (5) holds is characterized by a small parameter $\epsilon << 1$, where a balance between all terms requires the scaling $F \sim \epsilon^4$, $A \sim \epsilon^2$, $\partial/\partial x \sim \epsilon^4$, $\partial/\partial t \sim \epsilon^3$, and $\Delta \sim \epsilon^2$. Note that ζ scales with $\sqrt{F}$ and with the detuning Δ, a feature typical of forced resonant systems.

The canonical form of the fKdV equation,

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x(x) = 0, \tag{6}$$

is obtained by putting $\zeta = 2A/3$, $\Delta = \tilde{\Delta}/6$, $t = 6\tilde{t}$, $F = 2\tilde{F}/9$, and then omitting the "tilde." Equation (6) is typically solved with the initial condition that $A(x, 0) = 0$, which corresponds to the introduction of the topographic obstacle $F(x)$ at $t = 0$.

4 Transcritical flow over a localized obstacle

Figures (1-3) are a series of solutions with varying criticality (Δ) for the canonical form of the fKdV equation where the forcing function is localized at $x = 0$ with a maximum height $F_M > 0$ and is "Gaussian" in shape. In these figures, F_M is specifically $F_M = 1$.

In Figures 1-3 it can be seen that solutions of the fKdV equation typically consist of upstream and downstream nonlinear wavetrains connected by a locally steady solution over the obstacle. The origin and behavior of the nonlinear wavetrains is due to the structure of the locally steady solution over the obstacle. Assuming that the dispersionless or "hydraulic" limit is applicable in the obstacle region, the linear dispersive term in equation (6) can be neglected. A result of this is that for all localized $F(x)$ with a maximum height $F_M > 0$,

$$6A_{\pm} = \Delta \mp (12F_M)^{1/2}. \tag{7}$$

The transcritical regime is thus defined as

$$|\Delta| < (12F_M)^{1/2}. \tag{8}$$

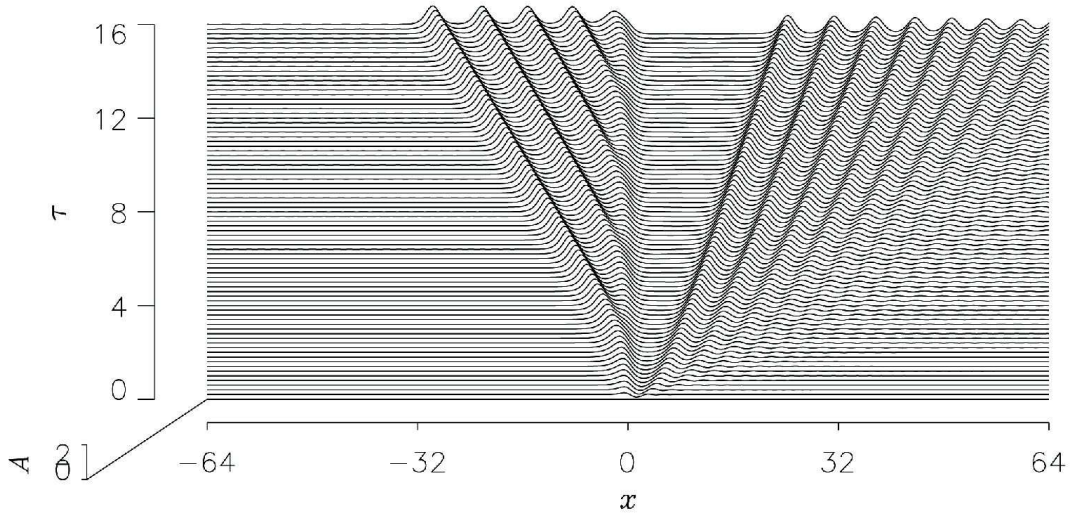

Figure 1: Solution to the fKdV equation (6) at exact criticality, $\Delta = 0$. The forcing (not shown in the plot) is located at $x = 0$ and has a maximum height of $F_M = 1$

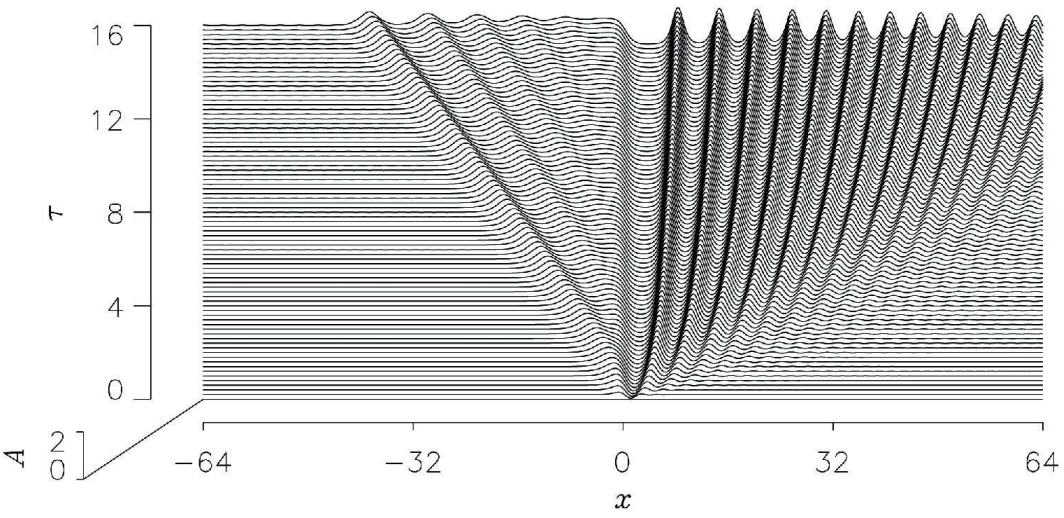

Figure 2: Solution to the fKdV equation (6) for a subcritical case, $\Delta = -1.5$.

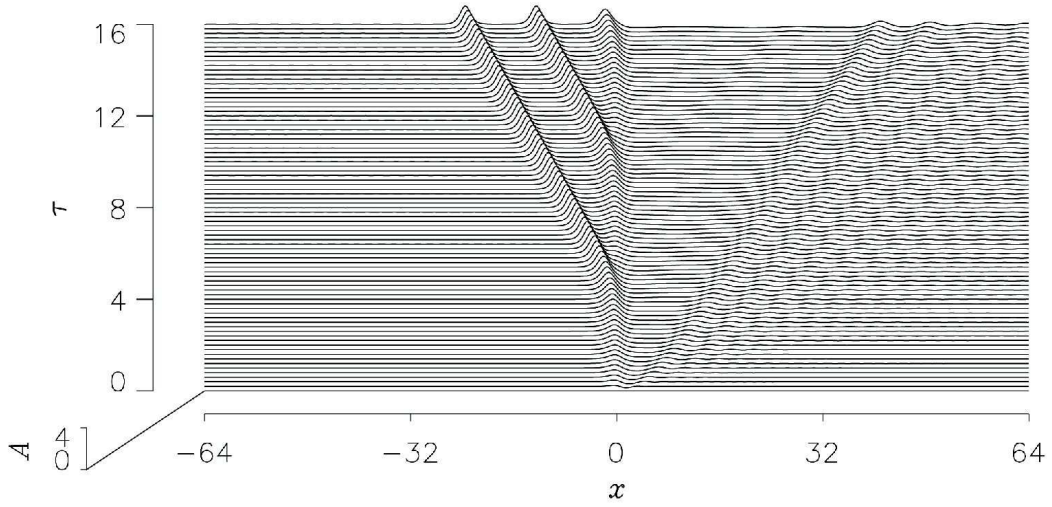

Figure 3: Solution to the fKdV equation (6) for a supercritical case, $\Delta = 1.5$.

In the transcritical regime, the local steady solution is characterized by a transition from a constant elevation $A_- > 0$ upstream ($x < 0$) of the obstacle to a constant depression $A_+ < 0$ downstream ($x > 0$) of the obstacle, where $\Delta = 3(A_+ + A_-)$, independent of the details of the localized forcing term $F(x)$. The explicit values of A_+ and A_- are determined by the forcing term $F(x)$, as shown in Sections 5.1 and 5.2. The upstream and downstream solutions are solved with undular bores.

4.1 Undular bores

A simple representation of an undular bore can be obtained from the solution of the KdV equation

$$A_t + 6AA_x + A_{xxx} = 0 \tag{9}$$

with the initial condition of a step, $A = A_0 H(-x)$ with $A_0 > 0$ and $H(x)$ the Heaviside function. An asymptotic solution can be found using Whitham's modulation theory. This formalism was presented in Lecture 10. The relevant asymptotic solution corresponding to the "step" initial condition is constructed in terms of a similarity variable x/t. The undular bore wavetrain is located in the region

$$-6 < \frac{x}{A_0 t} < 4. \tag{10}$$

Also recall that if the initial condition A_0 corresponds to a negative step $A_0 < 0$ then the undular bore solution does not exist. Instead the asymptotic solution is a rarefaction wave,

$$A = 0 \text{ for } x > 0,$$
$$A = \frac{x}{6t} \text{ for } A_0 < \frac{x}{6t} < 0,$$

$$A = A_0, \text{ for } \frac{x}{6t} < A_0 < 0. \tag{11}$$

Small oscillatory wave trains are needed to smooth out the discontinuities in A_x at $x = 0$ and $\frac{x}{t} = -6A_0$.

4.2 Asymptotic analysis for localized forcing

We now return to the asymptotic solution of the fKdV equation, and resolve the upstream and downstream transitions by the "undular bore" solutions. Making the appropriate transformations for the upstream wavetrain in (10) (see [2]), the upstream wavetrain ($x < 0$) occupies the region

$$\Delta - 4A < \frac{x}{t} < \min\{0, \Delta + 6A\}. \tag{12}$$

The upstream wavetrain cannot move beyond the obstacle (at $x = 0$), and thus is only fully realized if $\Delta < -6A_-$. Combining this criterion with (7) and (8) defines the regime

$$-(12F_M)^{1/2} < \Delta < -\frac{1}{2}(12F_M)^{1/2} \tag{13}$$

where a fully developed undular bore solution can develop upstream. On the other hand, when $\Delta > -6A_-$ or

$$-\frac{1}{2}(12F_M)^{1/2} < \Delta < (12F_M)^{1/2} \tag{14}$$

the undular bore cannot develop beyond the obstacle. A partially formed undular bore develops upstream, and this is attached to the obstacle.

The downstream wavetrain similarly is constrained to lie in $x > 0$. The undular bore solution occupies the zone

$$\max\{0, \Delta - 2A_+\} < \frac{x}{t} < \Delta - 12A_+ \tag{15}$$

The downstream wavetrain is only fully realized if $\Delta > 2A_+$. Combining this criterion with (7) and (8), the region where a fully developed downstream undular bore coincides with (14), the region where the upstream wavetrain is partially formed and attached to the obstacle. Similarly, when the downstream undular bore ($\Delta < 2A_+$) is only partially developed and attached to the obstacle, the upstream undular bore is fully developed and detached from the obstacle.

Figures (1-3) show this behavior in the regime defined by (14) where the upstream wavetrain is partially formed and attached to the obstacle while the downstream wavetrain is fully developed. In the subcritical case ($\Delta < 0$), the upstream wavetrain weakens and for sufficiently large $|\Delta|$ separates from the obstacle (i.e. makes the transition from the regime defined by (14) to the regime defined by (13)). When this transition is reached, the downstream wavetrain intensifies and forms a stationary lee wave field. For supercritical flow ($\Delta > 0$), the behavior is always governed by (14). As $|\Delta|$ increases the upstream wavetrain develops into well-separated solitary waves attached at the obstacle while the downstream wavetrain weakens and moves downstream.

For the case where the obstacle has negative polarity ($F_M < 0$), the upstream and downstream solutions are qualitatively similar (i.e. can be described in terms of an undular

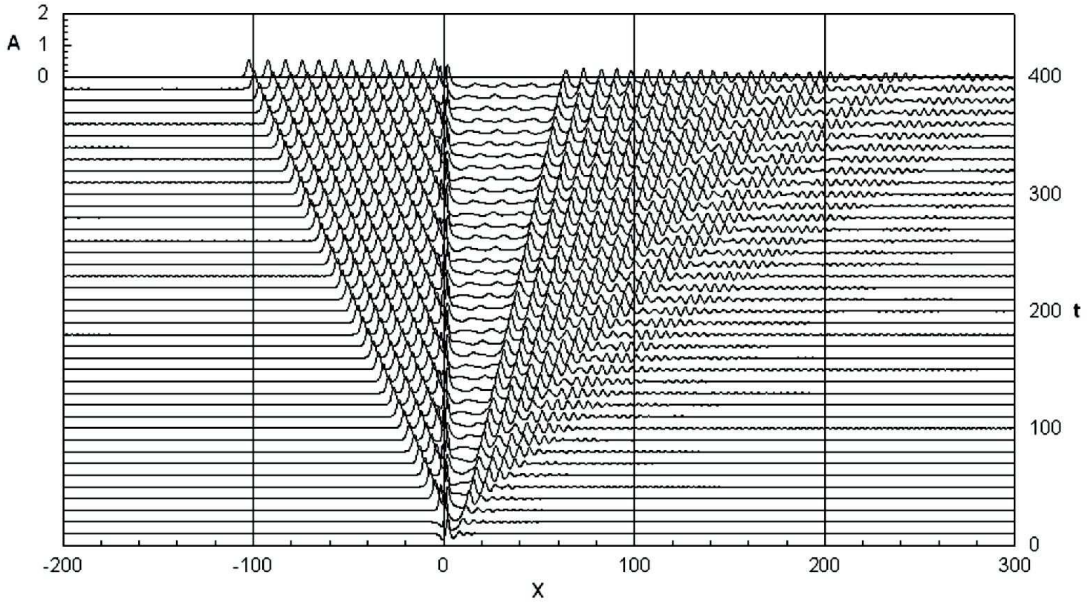

Figure 4: Solution to the fKdV equation (6) for exact criticality, $\Delta = 0$. The forcing (not shown in the plot) is located at $x = 0$ and has localized negative forcing $F_M = -0.1$.

bore solution). However, the local solution around the obstacle is transient, and this causes a modulation of the undular bore solutions seen previously. A typical solution of the fKdV equation with a negative polarity obstacle is shown in Figure (4). Note the perturbations in the solution around the obstacle.

5 Transcritical flow over a step

5.1 Asymptotic analysis for positive step forcing

First, consider a broad positive step, where

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ F_M & \text{for } x > W \end{cases} \tag{16}$$

and $F(x)$ varies monotonically in $0 < x < W$, and $F_M > 0$. Strictly $F(x)$ should return to zero for some $L >> W$. For this analysis, we ignore this and assume that $L \to \infty$. We revisit the step of finite length in the numerical simulations of Section 5.3. We now modify the asymptotic solution found for localized forcing in Section 4 for an infinitely long step. The first step is to construct the local steady-state solution, using the hydraulic limit. In the forcing region ($0 < x < W$), $A = A(x)$, while otherwise

$$A = A_- \text{ for } x \to -\infty, \tag{17}$$

$$A = A_+ \text{ for } x \to \infty. \tag{18}$$

The steady-state fKdV equation (6) ignoring the dispersive term is

$$-\Delta A_x + 6AA_x + F_x = 0. \tag{19}$$

Integrating (19) with respect to x results in the equation

$$-\Delta A + 3A^2 + F = C, \tag{20}$$

whose solutions are

$$6A = \Delta \pm (\Delta^2 + 12C - 12F)^{1/2}. \tag{21}$$

Thus, there are two branches to the solution. Applying the far-field limits (17, 18) to equation (20) yields

$$C = -\Delta A_- + 3A_-^2 = -\Delta A_+ + 3A_+^2 + F_M. \tag{22}$$

The constant C is obtained by taking the long-time limit of the unsteady hydraulic solution. The fKdV equation (6) with no dispersion is a nonlinear hyperbolic equation that can be solved by the the method of characteristics. This equation written in characteristic form is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Delta - 6A, \ \frac{\mathrm{d}A}{\mathrm{d}t} = F_x(x) \tag{23}$$

with the initial condition that $A = 0$ at $t = 0$. For a positive step, F_x is zero for $x < 0$ and $x > W$ and $F_x > 0$ for $0 < x < W$. From (23), all characteristics begin with an initial slope Δ and then decrease. The key issue is whether the characteristics ever reach a turning point ($\mathrm{d}x/\mathrm{d}t = 0$, $\Delta = 6A$).

For $\Delta \leq 0$, all characteristics have a negative slope and there are no turning points. In this case, $A_+ = 0$ and from (22), $C = F_M$ and the upper branch must be chosen in (21). For $\Delta > (12F_M)^{1/2}$, there are also no turning points, and all characteristics have positive slope. In this case, $A_- = 0$, $C = 0$ and the lower branch is chosen. For $0 < \Delta < (12F_m)^{1/2}$ there is a turning point. Characteristics emerging from the step ($0 < x < W$) with $F = F_0$, $0 < 12F_0 < 12F_M - \Delta^2$ have a turning point and then go upstream into $x < 0$. From this, $12C = 12F_M - \Delta^2$ and from (22), $6A_+ = \Delta$, while A_- is then obtained from the upper branch of (21). In summary, the upstream and downstream solutions for a positive step forcing are

$$\Delta \leq 0: \ 6A_- = \Delta + (\Delta^2 + 12F_M)^{1/2}, 6A_+ = 0, \tag{24}$$

$$0 < \Delta < (12F_M)^{1/2}: \ 6A_- = \Delta + (12F_M)^{1/2}, 6A_+ = \Delta, \tag{25}$$

$$\Delta > (12F_M)^{1/2}: \ 6A_- = 0, 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}. \tag{26}$$

In all cases, the upstream solution A_- is a "jump" in the hydraulic limit, and thus needs to be resolved by an undular bore. The upstream undular bore is located in the region

$$\Delta - 4A < \frac{x}{t} < \min\{0, \Delta + 6A\}, \tag{27}$$

the region similar to that for the upstream jump for a localized forcing (12). For a fully detached undular bore, $\Delta + 6A_- < 0$, and combining this criterion with (24, 25, 26), the defining regime is

$$\Delta < -2(F_M)^{1/2} < 0. \tag{28}$$

In the regime where $\Delta + 6A_- > 0$ but $\Delta - 4A_- < 0$, or

$$-2(F_M)^{1/2} < \Delta < (12F_M)^{1/2}, \tag{29}$$

the upstream undular bore is only partially formed and is attached to the positive step. Unlike the localized obstacle, the downstream profile A_+ is not a jump ($A_+ > 0$). The solution downstream is terminated instead by a rarefaction wave.

5.2 Asymptotic analysis for negative step forcing

A similar analysis to Section 5.1 for a positive step can be carried out for a negative step ($F_M < 0$). The results are

$$\Delta \geq 0: \ 6A_- = 0, 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2} \text{ with } C = 0, \tag{30}$$

$$-(|12F_M|)^{1/2} < \Delta < 0: \ 6A_- = \Delta, 6A_+ = \Delta - (|12F_M|)^{1/2} \text{ with } C = -\Delta^2/12, \tag{31}$$

$$\Delta < -(|12F_M|)^{1/2}: \ 6A_- = \Delta - (\Delta^2 - 12F_M)^{1/2}, 6A_+ = 0 \text{ with } C = F_M. \tag{32}$$

In all cases, the downstream solution A_+ in (30-32) is negative and contains a jump. This jump needs to be resolved by an undular bore, occupying the zone similar to (13)

$$\max\{0, \Delta - 2A_+\} < \frac{x - W}{t} < \Delta - 12A_+. \tag{33}$$

For a fully detached undular bore ($\Delta - 2A_+ > 0$) is obtained in the regime

$$\Delta > -(|3F_M|)^{1/2}. \tag{34}$$

The downstream undular bore is partially attached when $\Delta - 2A_+ < 0$ corresponding to a regime where

$$-(|12F_M|)^{1/2} < \Delta < -(|3F_M|)^{12} < 0. \tag{35}$$

A stationary lee-wave train forms downstream when $\Delta < -(|12F_M|)^{1/2}$. The upstream solution A_- is less than zero, and thus terminated by a rarefaction wave.

5.3 Numerical solutions for the fKdV equation with a step forcing

Combining the results from Sections 5.1 and 5.2, we can now understand the behavior of flow over a finite-length positive step forcing; that is, a step up, followed by a region of constant elevation and terminated by a step down. Figure 5 shows numerical simulations of the fKdV equation for water waves (5) with varying Δ for a step forcing of the form

$$F(x) = \frac{F_M}{2}(\tanh \gamma x - \tanh \gamma(x - L)), \tag{36}$$

where $F_M = 0.1$, $\gamma = 0.25$ and $L = 50 >> 1$. Note these are parameter values for the unscaled parameters in the water wave formulation.

When the flow is critical ($\Delta = 0$), the theory predicts an upstream undular bore attached to the obstacle (regime given by (29)) and a fully detached downstream undular bore that

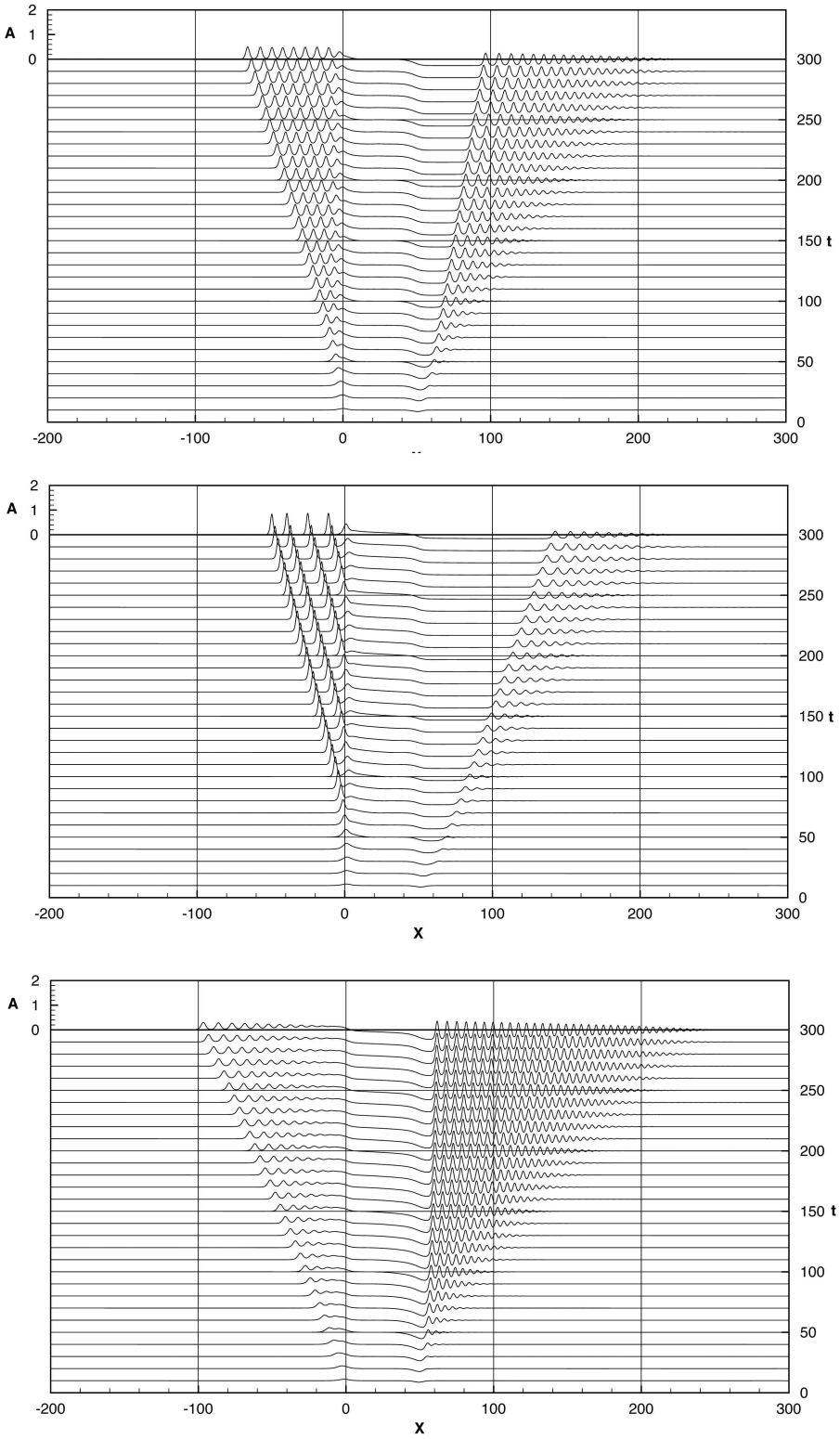

Figure 5: Numerical simulations of the fKdV equation (5) for the forcing (36), where $F_M = 0.1$, $\gamma = 0.25$, $L = 50 \gg 1$ and (a)$\Delta = 0.$, (b)$\Delta = 0.2$, (c)$\Delta = -0.2$. Note, to obtain the correct boundary values given by (28, 29) and (34, 35), the parameters must be scaled to those for the canonical fKdV (6).

propagates downstream (34). This is seen in Figure 5a, and is similar to the behavior of exactly resonant flow over a localized obstacle (Figure 1).

The behavior of a supercritical flow is shown in Figure 5b. Once again, the upstream undular bore is only partially formed and the downstream undular bore is fully developed and propagates downstream. The upstream solution however has transitioned from the solution defined by (24) to that defined by (25); the downstream solution still remains in the regime given by (34). A rarefaction wave thus propagates downstream from the positive step at $x = 0$. The rarefaction wave persists until it reaches the end of the step at $x = 50$, and the solution adjusts to a solution predicted for simply a localized forcing (one upstream undular bore and one downstream undular bore).

Figure 2c shows a numerical simulation for subcritical flow where $\Delta = -0.2$. In this case, there is a detached undular bore propagating upstream and a detached downstream bore that intensifies and propagates slowly. The parameter values in this simulation are close to the boundaries predicted by the asymptotic theory, and while the upstream bore falls in the regime (29) given for an attached undular bore, the behavior in the simulation suggests that the upstream solution actually follows regime (28). Similarly, the downstream bore should fall into regime (35), but actually follows regime (34). This is attributed to errors in the estimates of $A_{\mp}$ and the resulting boundaries derived from those quantities. Note that the downstream solution is in the regime defined by 31 and thus a rarefaction wave propagating upstream is generated at the negative step ($x = 50$).

5.4 Comparison of the fKdV solutions to solutions of the full Euler equations

Figure 6 shows numerical simulations of the full Euler equations for the same positive step forcing as before. In these simulations, there seems to be good qualitative agreement with the behavior of the solutions of the fKdV equation. Table 1 shows the quantitative differences between the simulations. The amplitudes of the leading upstream and downstream waves are consistently larger in the fKdV simulations likely due to nonlinearity. The variation of all the predicted amplitudes and elevations as Δ is varied follows the same trend for both the fKdV and Euler equations. Furthermore, in all of the Euler equation simulations, there were no other wavetrains generated than those seen in the simulations of the fKdV equation. Therefore, for small-amplitude steps, the fKdV equation likely provides a good guide for transcritical flow over a step.

5.5 Solutions to a negative step forcing

Similar numerical simulations can be carried out for an obstacle where the step forcing in (36) is negative, $F_M < 0$. The flow first encounters a negative step, followed by a constant depression section, and is terminated by a positive step. The asymptotic analyses from Sections 5.1 and 5.2 suggest that the resulting undular bore propagates downstream of the negative step and another undular bore similarly propagates upstream of the positive step. These undular bores interact with each other over the step itself.

In the depression region, the interactions between the two undular bores can be quite complex as seen in Figures 7-8. In Figure 7, the undular bores pass through each other

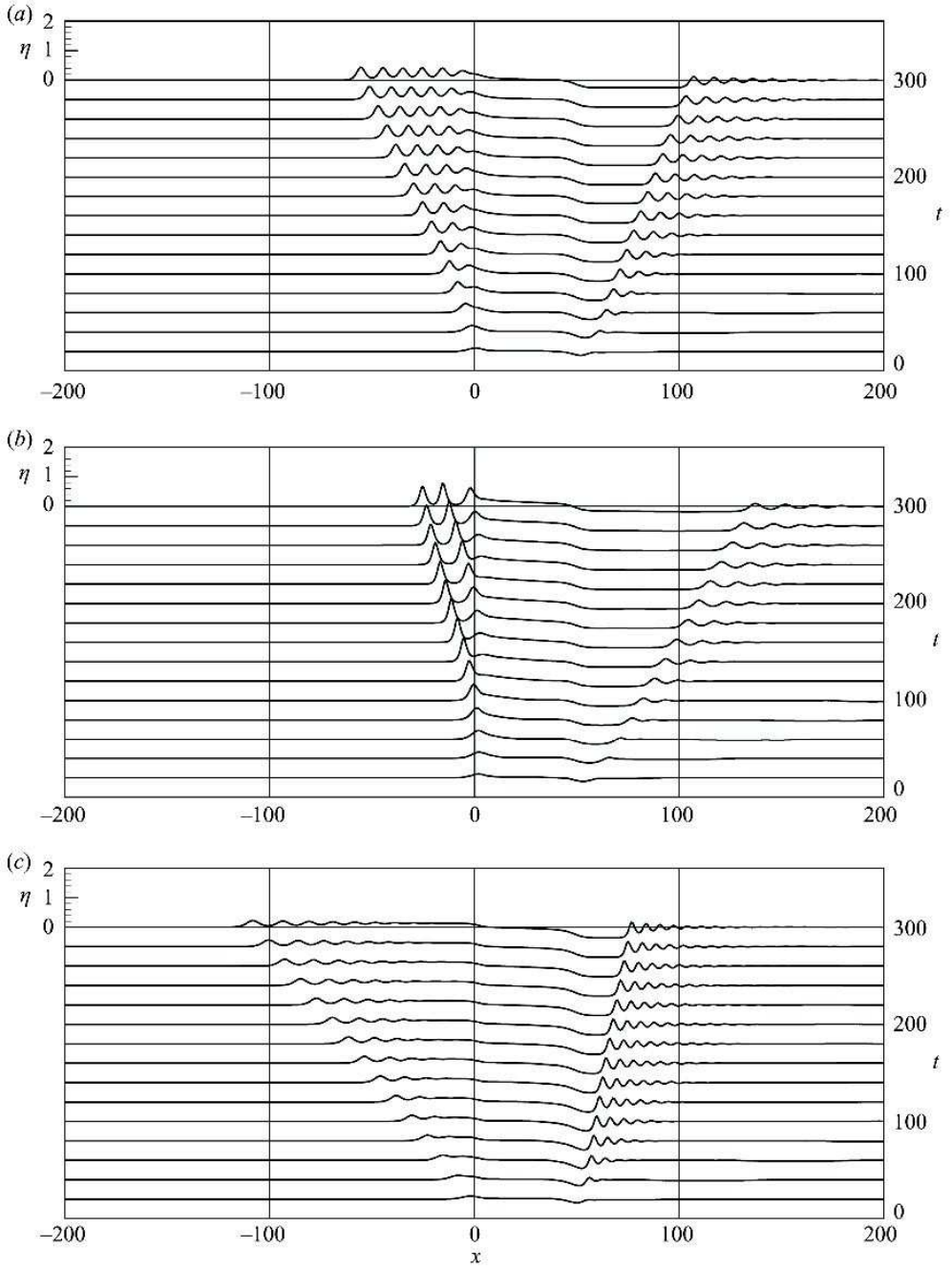

Figure 6: Numerical simulations of the full Euler equations for the forcing (36), where $F_M = 0.1$, $\gamma = 0.25$, $L = 50 \gg 1$ and (a)$\Delta = 0.$, (b)$\Delta = 0.2$, (c)$\Delta = -0.2$.

Δ	fKdV				Euler			
	A_{W-}	A_-	A_{W+}	A_+	A_{W-}	A_-	A_{W+}	A_+
0.2	0.83	0.44	0.31	-0.16	0.75	0.40	0.28	-0.18
0.1	0.66	0.38	0.39	-0.20	0.57	0.36	0.32	-0.21
0.0	0.50	0.30	0.51	-0.26	0.44	0.33	0.37	-0.25
-0.1	0.39	0.22	0.64	-0.33	0.32	0.20	0.43	-0.30
-0.2	0.30	0.16	0.84	-0.40	0.23	0.13	0.53	-0.36
-0.3	0.24	0.13	0.64	-0.38	0.16	0.08	0.57	-0.38
-0.4	0.19	0.10	0.00	0.00	0.10	0.01	0.00	0.00

Table 1: Quantitative comparison of the results from the fKdV equation (5) and the Euler equations. $A_-(A_+)$ is the elevation just upstream (downstream) of the positive (negative) step at $x = 0(50)$ respectively, and $A_{W-}(A_{W+})$ is the amplitude of the leading wave in the corresponding undular bore.

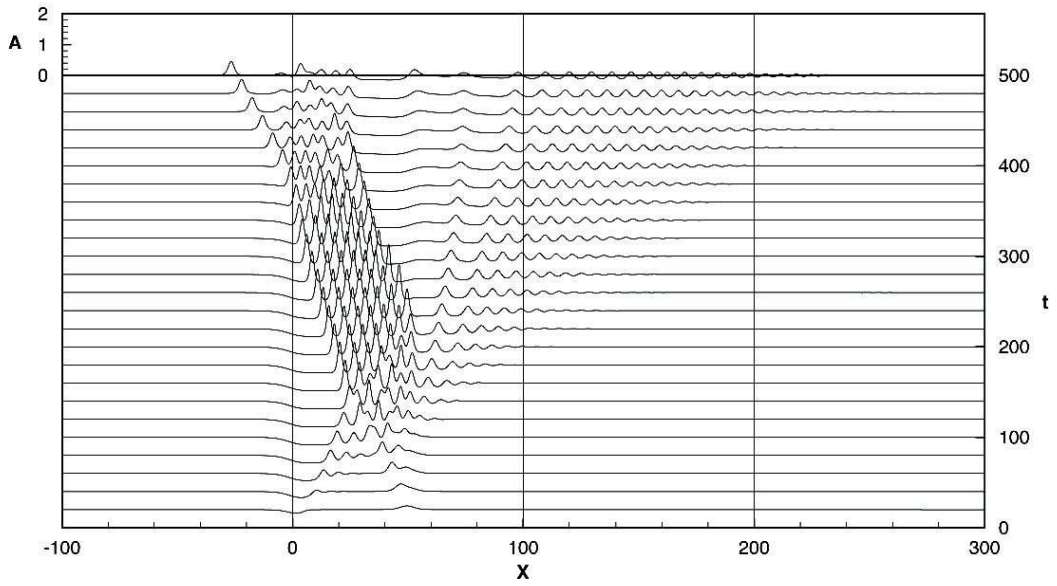

Figure 7: Numerical simulation of the fKdV equation (6) for the step forcing (36) where $F_M = -0.1$, $\gamma = 0.25$, $L = 50 \gg 1$ and $\Delta = 0.0$.

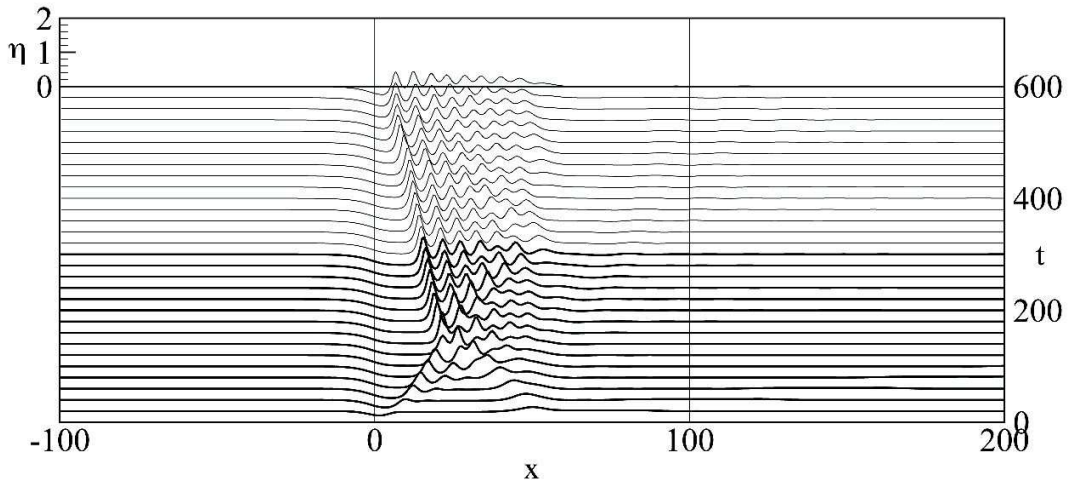

Figure 8: Numerical simulation of the full Euler equations for the step forcing (36) where $F_M = -0.1$, $\gamma = 0.25$, $L = 50 \gg 1$ and $\Delta = -0.2$.

with interference in the depression region. For moderate time, both the upstream and downstream solutions are fairly unperturbed by each other.

Figure 7 shows a solution of the fKdV equation to a negative step forcing, while Figure 8 shows a solution to the full Euler equations for the same forcing. For the positive step, it seemed that there were no qualitative differences between the fKdV solutions and the solutions to the full Euler equations. However, the differences in Figures 7 and 8 are significant. In Figure 8 the wave propagation is largely restricted to the depression region. The waves in the full Euler solutions are likely confined to the step region due to the interaction of two effects not accounted for in the fKdV equation. First, the steps have a finite amplitude that the wave must pass over in order to propagate. Small amplitude waves may reflect at the steps, instead of propagating forward. In addition, the depression also causes a depression in the local wave speed in the step region, and waves may have lower amplitude due to this effect.

References

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