Lecture 14: Waves on deep water, I

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1 Introduction

In this lecture we address the question of whether there are stable wave patterns that propagate with permanent (or nearly permanent) form on deep water. The primary tool for this investigation is the nonlinear Schrödinger equation (NLS). Below we sketch the derivation of the NLS for deep water waves, and review earlier work on the existence and stability of 1D surface patterns for these waves. The next lecture continues to more recent work on 2D surface patterns and the effect of adding small damping.

2 Derivation of NLS for deep water waves

The nonlinear Schrödinger equation (NLS) describes the slow evolution of a train or packet of waves with the following assumptions:

- the system is conservative (no dissipation)
- then the waves are dispersive (wave speed depends on wavenumber)

Now examine the subset of these waves with

- only small or moderate amplitudes
- traveling in nearly the same direction
- with nearly the same frequency

The derivation sketch follows the by now normal procedure of beginning with the water wave equations, identifying the limit of interest, rescaling the equations to better show that limit, then solving order-by-order.

We begin by considering the case of only gravity waves (neglecting surface tension), in the deep water limit ($kh \to \infty$). Here h is the distance between the equilibrium surface height and the (flat) bottom; a is the wave amplitude; η is the displacement of the water surface from the equilibrium level; and ϕ is the velocity potential, $\mathbf{u} = \nabla\phi$. (See also Lecture 4.) In terms of our small parameter ϵ $(= ah)$, we will consider waves that are nearly monochromatic:

$$\begin{aligned} \vec{k} &= (k_0, 0) + O(\epsilon) \\ \omega &= \sqrt{gk_0} + O(\epsilon) \end{aligned}$$

and small in amplitude:

$$k_0\|\eta\| = O(\epsilon)$$

We use a coordinate $\theta = k_0 x - \omega(k_0)t$ and a solution form

$$\eta(x, y, t; \epsilon) = \epsilon[A(\epsilon x, \epsilon y, \epsilon t, \epsilon^2 t)e^{i\theta} + A^* e^{-i\theta}] + \epsilon^2[\text{stuff}_2] + \epsilon^3[\text{stuff}_3] + O(\epsilon^4)$$

and then insert the formal expansions for $\eta(x, y, t; \epsilon)$ and $\phi(x, y, t; \epsilon)$ into the full equations.

Solving order by order, the algebra is nasty, but mostly avoidable via one of the usual suspects such as maple. What we find is the following:

- At $O(\epsilon)$, $\omega^2 = gk$, we recover the linearized dispersion relation for gravity-induced waves on deep water.
- At $O(\epsilon^2)$, the expansion blows up and becomes disordered unless

  $$\frac{\partial A}{\partial(\epsilon t)} + c_g \frac{\partial A}{\partial(\epsilon x)} = 0$$

  where c_g is the group velocity (the propagation speed of the wave envelope).

In η above, now define a coordinate moving with the wave $\xi = (\epsilon x) - c_g(\epsilon t)$, a slowly varying transverse direction $\zeta = \epsilon y$, and a slowly varying time $\tau = \epsilon^2 t$. Now examine $O(\epsilon^3)$, where the expansion again becomes disordered unless $A(\xi, \zeta, \tau)$ satisfies

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma|A|^2 A = 0 \tag{1}$$

which is the 2D nonlinear Schrödinger equation. Here α, β, and γ are real numbers determined by the problem. For the case of deep water waves, the signs of the coefficients (independent of scaling choice) are $\alpha < 0$, $\beta > 0$, and $\gamma < 0$.

The NLS or some equivalent has been derived at many times and in many contexts, including: Zahkarov in 1968 for water waves [14], Ostrovsky in 1967 for optics [8], Benjamin and Feir in 1967 for water waves [3], a general formulation by Benney and Newell in 1967 [4], Whitham in 1965 [12], whose formulation was used by Lighthill in 1965 [6], and finally by Stokes in 1847 for water waves with no spatial dependence. A historical overview is provided by Zakharov and Ostrovsky [15].

3 Waves of permanent form on deep water

Now we come to the overarching question of interest in this and the next lecture: do stable waves of permanent form in deep water exist? Starting back at the beginning, Stokes [9] considered a spatially uniform train of plane waves,

$$\eta(x, y, t; \epsilon) = \epsilon[A(\epsilon^2 t)e^{i\theta} + A^* e^{-i\theta}] + O(\epsilon^2)$$

which after removing the spatial derivatives from (1) must satisfy

$$i\partial_\tau A + \gamma|A|^2 A = 0\ ,\ \gamma = -4k_0^2$$

where the expression for γ depends on the choice of scalings (see [1]). If we take the complex conjugate of this equation, multiply the original by A^* and the conjugate by A, and subtract them, we find:

$$iA^*\partial_\tau A + iA\partial_\tau A^* = 0$$

In other words, $\partial_\tau(A^*A) = 0$, so the square of the amplitude is a constant. With that, the equation is easy to solve, yielding

$$A(\tau) = (A_0 e^{i\phi})e^{i\gamma|A_0|^2\tau}$$

Putting this into the η expression above, and using θ from earlier, we have

$$\eta(x,t;\epsilon) = 2\epsilon|A_0|\cos\left[k_0x - \omega(k_0)t - (2\epsilon k_0|A_0|)^2 t\right] + O(\epsilon^2)$$

where the final term inside the cosine is Stokes' nonlinear correction to the frequency.

Although Stokes found a nonlinear correction for water waves of permanent form with finite amplitude, he did not prove that such waves existed. Nekrassov (available in [7]) and Levi-Civita [5] accomplished this task in the 1920s, and Struik [10] extended their work from deep water to water of any constant depth. Finally, Amick and Toland [2] obtained optimal results about the existence of waves of permanent form in 2D. The solutions in question, a periodic train of plane waves, are approximated by experimental data such as in the top panel of Figure 1.

3.1 Why don't we see them?

Despite the above work to prove the existence of such waves, they are not commonly observed in nature, but perhaps they can be produced in more controlled conditions. Photos from Benjamin in 1967, in Figure 1, show a wave train disintegrating in 60 meters, as discussed in the stability analysis of [3].

Returning to the NLS in Zakharov 1968,

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma|A|^2 A = 0$$

for the case of gravity waves on deep water, we have $\alpha < 0$, $\beta > 0$, and $\gamma < 0$. The "Stokes wave" above is the solution for a spatially uniform, finite amplitude train of plane waves to third order expansion in the water wave equations. We can now check its stability by linearizing the NLS around a Stokes wave and looking for unstable modes.

$$A(\xi,\zeta,\tau) = |A_0|\, e^{i\gamma|A_0|^2\tau}$$

Assume additional small perturbations:

$$A(\xi,\zeta,\tau) = e^{i\gamma|A_0|^2\tau}[|A_0| + \mu\cdot u(\xi,\zeta,\tau) + i\mu\cdot v(\xi,\zeta,\tau)] + O(\mu^2)$$

Putting this into (1), above, and keeping only terms up to $O(\mu)$, we have:

$$\begin{aligned} &e^{i\gamma|A_0|^2\tau}\left[-\gamma|A_0|^2\left(|A_0| + \mu u + i\mu v\right) + i\mu(\partial_\tau u + i\partial_\tau v)\right.\\ &\quad\left.+\alpha\mu(\partial_\xi^2 u + i\partial_\xi^2 v) + \beta\mu(\partial_\zeta^2 u + i\partial_\zeta^2 v) + \gamma|A_0|^2(|A_0| + \mu u + i\mu v) + 2\gamma|A_0|^2\mu u\right] = 0\end{aligned}$$

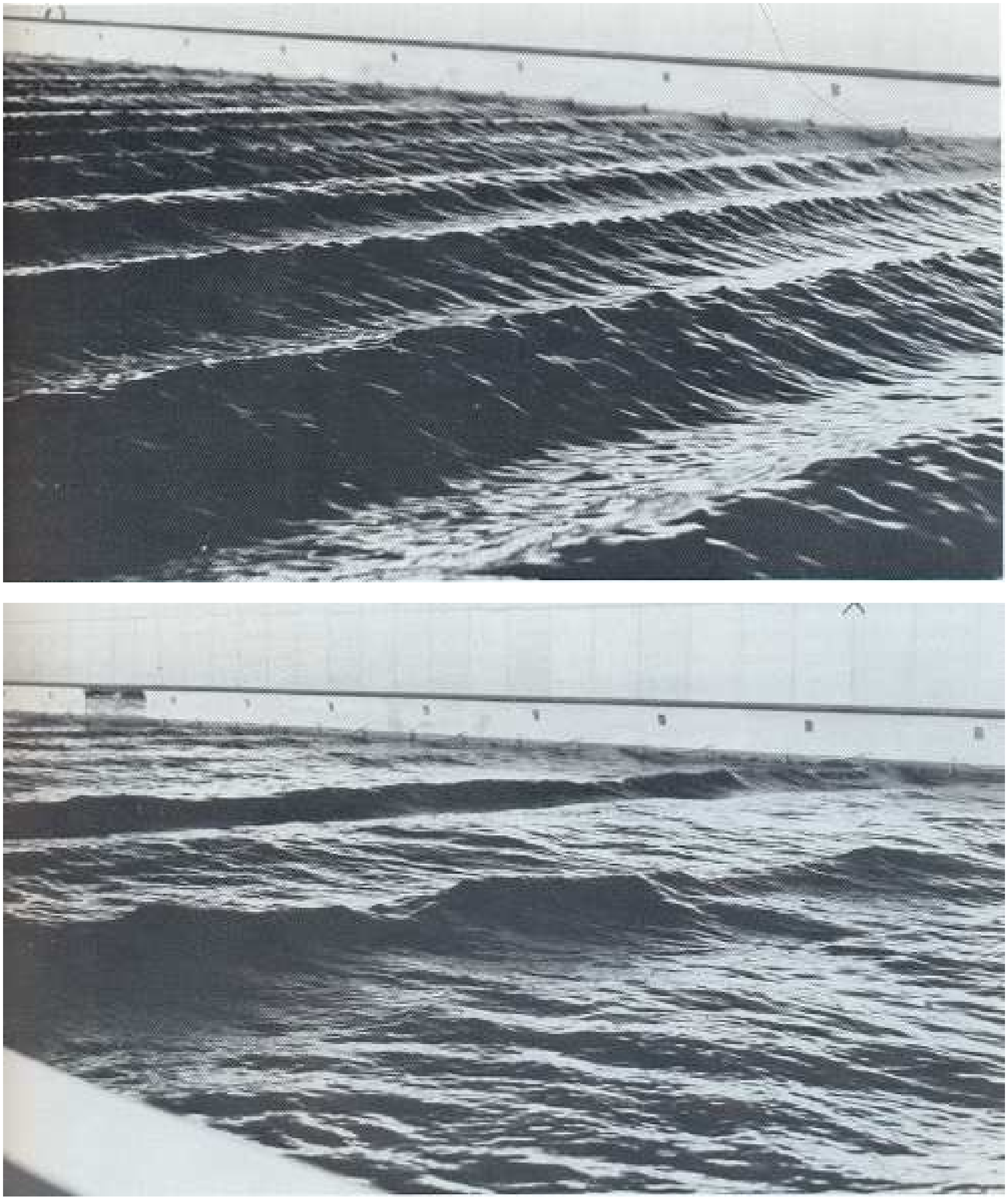

Figure 1: Wavetrain in deep water ($L = 2.3$ m, $h = 7.6$ m), with 60 meters between photos.

where the real and imaginary parts respectively break into a pair of linear PDEs with constant coefficients,

$$\partial_\tau v = \alpha\partial_\xi^2 u + \beta\partial_\zeta^2 u + 2\gamma|A_0|^2 u \tag{2}$$

$$-\partial_\tau u = \alpha\partial_\xi^2 v + \beta\partial_\zeta^2 v \tag{3}$$

We seek a solution of the form

$$\begin{aligned} u &= U \cdot e^{im\xi + il\zeta + \Omega\tau} + (\text{c.c.}) \\ v &= V \cdot e^{im\xi + il\zeta + \Omega\tau} + (\text{c.c.}) \end{aligned}$$

Putting these into (2) and (3), above, and eliminating U and V, we are left with an algebraic equation for the linear stability:

$$\Omega^2 + (\alpha m^2 + \beta l^2)(\alpha m^2 + \beta l^2 - 2\gamma|A_0|^2) = 0 \tag{4}$$

where $Re(\Omega) > 0$, positive real growth rate, indicates linear instability. Marginal stability occurs at

$$\Omega = \pm\sqrt{-(\alpha m^2 + \beta l^2)(\alpha m^2 + \beta l^2 - 2\gamma|A_0|^2)} = 0$$

for which the first root (recalling that here $\alpha < 0$, $\beta > 0$) is

$$\begin{aligned} \alpha m^2 &= -\beta l^2 \\ l &= \pm\sqrt{\frac{|\alpha|}{\beta}}\, m \end{aligned}$$

which defines a pair of lines with opposite slope that cross at the origin. The second root is

$$\alpha m^2 + \beta l^2 - 2\gamma|A_0|^2 = 0$$

which defines a hyperbola. See Figure 2 for the regions described by this expression; instability occurs for points along the m-axis between the two sets of curves. (This can be seen by considering $l^2 = 0$ and m^2 small, for which $\alpha m^2 < 0$ and $\alpha m^2 - 2\gamma|A_0|^2 > 0$; similarly, a positive real root does not exist for $m^2 = 0$ and $l^2 \neq 0$.)

The maximum value of the growth rate can be easily determined by noting that l and m appear only in the combination $\alpha m^2 + \beta l^2$, so we can call that something easy like s and look for extrema of Ω^2:

$$\begin{aligned} \Omega^2 &= -s(s - 2\gamma|A_0|^2) \\ \frac{\partial\Omega^2}{\partial s} &= -(s - 2\gamma|A_0|^2) - s \\ 0 &= -2s_{max} + 2\gamma|A_0|^2 \\ s_{max} &= \gamma|A_0|^2 \end{aligned}$$

The growth rate evaluated at this point is

$$\begin{aligned} \Omega^2(s_{max}) &= -\gamma|A_0|^2(\gamma|A_0|^2 - 2\gamma|A_0|^2) \\ \Omega_{max} &= |\gamma|\,|A_0|^2 \end{aligned}$$

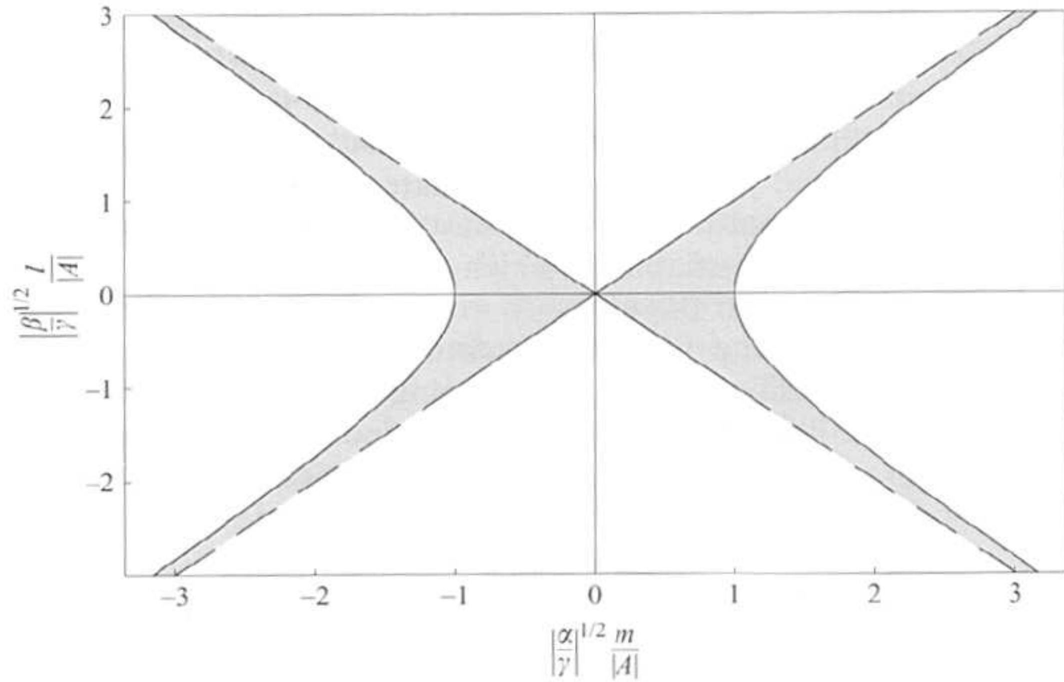

Figure 2: The regions of stability described by equation 4. Shaded regions are unstable.

Collecting the above, we have the result that a uniform train of finite amplitude plane waves is unstable in deep water, with the most unstable mode growing at $\Omega_{max} = |\gamma|\,|A_0|^2$. The instability is nonlinear in the sense that its growth rate depends on the amplitude $|A_0|$, so as $|A_0| \to 0$ then $\Omega_{max} \to 0$.

For applications other than deep-water gravity waves, we can repeat the above procedure for the stability diagram, where the respective signs of the coefficients will determine the existence of unstable regions. In general we have:

- $\alpha\beta < 0,\ \alpha\gamma > 0 \to$ unstable
- $\alpha\beta < 0,\ \alpha\gamma < 0 \to$ unstable
- $\alpha\beta > 0,\ \alpha\gamma > 0 \to$ unstable
- $\alpha\beta > 0,\ \alpha\gamma < 0 \to$ stable

One example of the instability of a uniform wavetrain was shown in Figure 1. For a second example in an electromagnetic context, see Figure 3 (which can be viewed as a cut across the center of the stability diagram, with the most unstable modes growing at either side).

3.2 A different wave pattern of permanent form

Having addressed the plane wave set of solutions above, and found them to be linearly unstable, another class of solutions is now our only hope for finding permanent stable wave

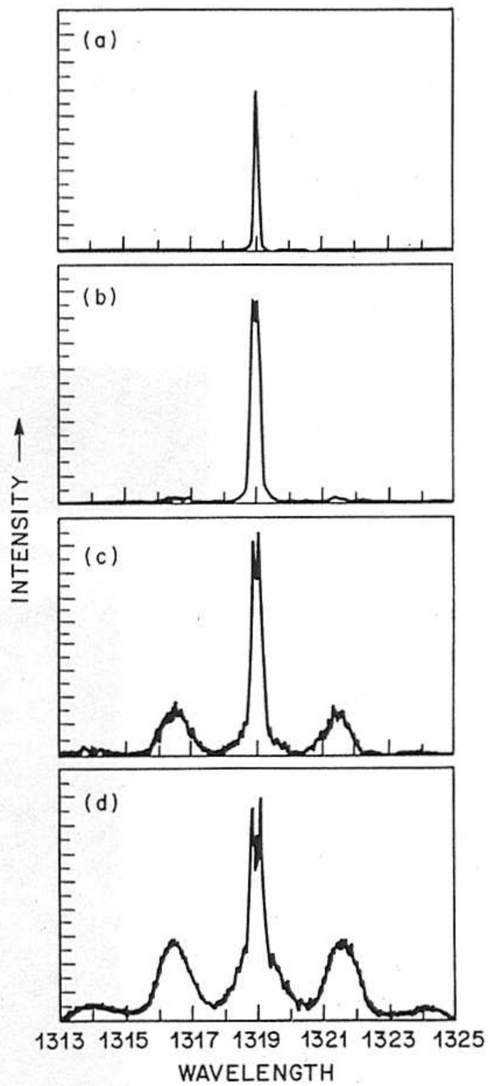

Figure 3: Experimental observation of modulational instability. The horizontal axis is in nanometers, so the length scale is $L = 1.3 \cdot 10^{-6}$ m, with timescale $T = 4 \cdot 10^{-15}$ s. Input power level low (a); 5.5 W (b); 6.1 W (c); 7.1 W (d). Figure adapted from [11].

patterns on deep water. This section will take a slightly different tack, using the Inverse Scattering Transform to solve the initial value problem (see Lecture 5) for the 1D NLS equation. This method enables us to show the existence of soliton solutions (which are by definition waves of permanent form) and deferring the question of their stability to transverse perturbations until the next lecture.

Continuing the search for another stable wave form, Zakharov and Shabat in 1972 [16] considered the NLS in 1D,

$$i\partial_\tau A = \partial_\xi^2 A + 2\sigma|A|^2 A \tag{5}$$

where $\sigma = -1$ is the "defocusing" case and $\sigma = +1$ is the "focusing" case appropriate for deep water. Setting $\sigma = 1$ and looking for traveling waves of "permanent form," there is a special case:

$$A(\xi,\tau) = 2a \cdot e^{-i(2a)^2\tau}\,\mathrm{sech}\,2a(\xi+\xi_0)$$

The corresponding shape of the free surface, as seen in Section 2, takes the form $\eta(x,t;\epsilon) = \epsilon(Ae^{i\theta} + A^* e^{-i\theta}) + O(\epsilon^2)$ where $\theta = kx - \omega(k)t$ so that

$$\eta(x,t;\epsilon) = (2\epsilon a)\,\mathrm{sech}\left[(2\epsilon a)(x - c_g t)\right]\cos\left\{kx - [\omega(k) + (2\epsilon a)^2]t\right\} \tag{6}$$

which is a wave packet with special shape (see Figure 4) discussed further below. This solution is reminiscent of the sech^2 soliton solutions of the KdV. This remark naturally leads us to ask which other properties of the KdV, and associated solution techniques, apply to the 1D NLS.

Zakharov and Shabat (1972) [16] demonstrated that the 1D NLS (5) is also completely integrable for either sign of σ. As for the KdV, there are an infinite number of explicit, local conservation laws, of which the first three are

$$\begin{aligned} i\partial_\tau(|A|^2) &= \partial_\xi(A^*\partial_\xi A - A\partial_\xi A^*) \\ i\partial_\tau(A^*\partial_\xi A - A\partial_\xi A^*) &= \partial_\xi(\ldots) \\ i\partial_\tau(|\partial_\xi A|^2 + \sigma|A|^4) &= \partial_\xi(\ldots) \end{aligned}$$

The other major consequence of complete integrability (see notes for Lecture 5) is that we can treat the problem with the inverse scattering transform. However, this time the scattering problem is no longer defined by the time-invariant Schrödinger equation, but is found instead to be given by the set of equations

$$\begin{aligned} \partial_\xi v_1 &= -i\lambda v_1 + A v_2 \\ \partial_\xi v_2 &= -\sigma A^* v_1 + i\lambda v_2. \end{aligned}$$

The requirement that the eigenvalues be independent of time, together with the compatibility condition

$$\partial_\xi\partial_\tau \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \partial_\tau\partial_\xi \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

recover the 1D NLS provided

$$\begin{aligned} \partial_\tau v_1 &= (-2i\lambda^2 + i\sigma|A|^2)v_1 + (2A\lambda + i\partial_\xi A)v_2 \\ \partial_\tau v_2 &= (-2\sigma A^*\lambda + i\sigma\partial_\xi A^*)v_1 - (-2i\lambda^2 + i\sigma|A|^2)v_2 \end{aligned}$$

Indeed,

$$\begin{aligned}
\partial_\xi \partial_\tau v_1 &= (i\sigma A \partial_\xi A^* + i\sigma A^* \partial_\xi A) v_1 + (-2i\lambda^2 + i\sigma |A|^2)(-i\lambda v_1 + A v_2) \\
&\quad + (2\lambda \partial_\xi A + i\partial_\xi^2 A) v_2 + (2A\lambda + i\partial_x iA)(-\sigma A^* v_1 + i\lambda v_2) \\
&= (i\sigma A \partial_\xi A^* - 2\lambda^3 - \sigma\lambda |A|^2) v_1 + (i\sigma |A|^2 A + \lambda \partial_\xi A + i\partial_\xi^2 A) v_2 \\
\partial_\tau \partial_\xi v_1 &= -i\lambda[(-2i\lambda^2 + i\sigma |A|^2) v_1 + (2A\lambda + i\partial_\xi A) v_2] + (\partial_\tau A) v_2 \\
&\quad + A[(-2\sigma A^* \lambda + i\sigma \partial_\xi A^*) v_1 + (2i\lambda^2 - i\sigma |A|^2) v_2] \\
&= (-2\lambda^3 - \sigma\lambda |A|^2 + i\sigma A \partial_\xi A^*) v_1 + (\lambda \partial_\xi A + \partial_\tau A - i\sigma |A|^2 A) v_2
\end{aligned}$$

Equating the two, all v_1 terms cancel, but for the two v_2 sides to balance, we must have

$$\partial_\tau A = i\partial_\xi^2 A + 2i\sigma |A|^2 A$$

Repeating the process for the v_2 equations produces the complex conjugate of the above. From this we arrive back at the 1D NLS,

$$i\partial_\tau A = \partial_\xi^2 A + 2\sigma |A|^2 A$$

as should be the case.

As a consequence of constructing this Inverse Scattering Transform method for the 1D NLS, it can be shown that for $\sigma = 1$ (focusing NLS), any smooth initial data $A(\xi, 0)$ with $\int |A| d\xi < \infty$ evolves into N "envelope solitons" which persist forever, plus an oscillatory wavetrain that decays in amplitude as $\tau \to \infty$. Envelope solitons are stable for the focusing case in the 1D NLS.

Figure 4 (from unpublished work by Hammack) shows experimental evidence for the existence of these envelope solitons, and reveal that they match the sech solution (6) very well. Figure 5 shows data from three other experiments on envelope solitons: the first shows the evolution of two solitons in isolation; the second has some additional noise that eventually separates from the soliton; the third shows one of the wave pulses overtaking and passing through another, emerging on the other side in (mostly) unchanged form. In contrast to Figure 1, where the uniform train of plane waves disintegrates quickly, the experiments in Figures 4 and 5 show the relative persistence of soliton solutions, providing some confirmation of the stability predicted by this 1D theory.

4 Conclusions

Finally, we reach the point of drawing tentative conclusions. According to the 1D or 2D NLS, a uniform train of plane waves is unstable in deep water. However, according to focusing NLS in 1D with initial data in L_1 (i.e., meeting the integrability condition above), envelope solitons are stable in deep water. Experimental evidence seems to support both of these conclusions, but we have not yet addressed the stability of the 1D soliton solutions to transverse perturbations. For more on that topic, see the next lecture.

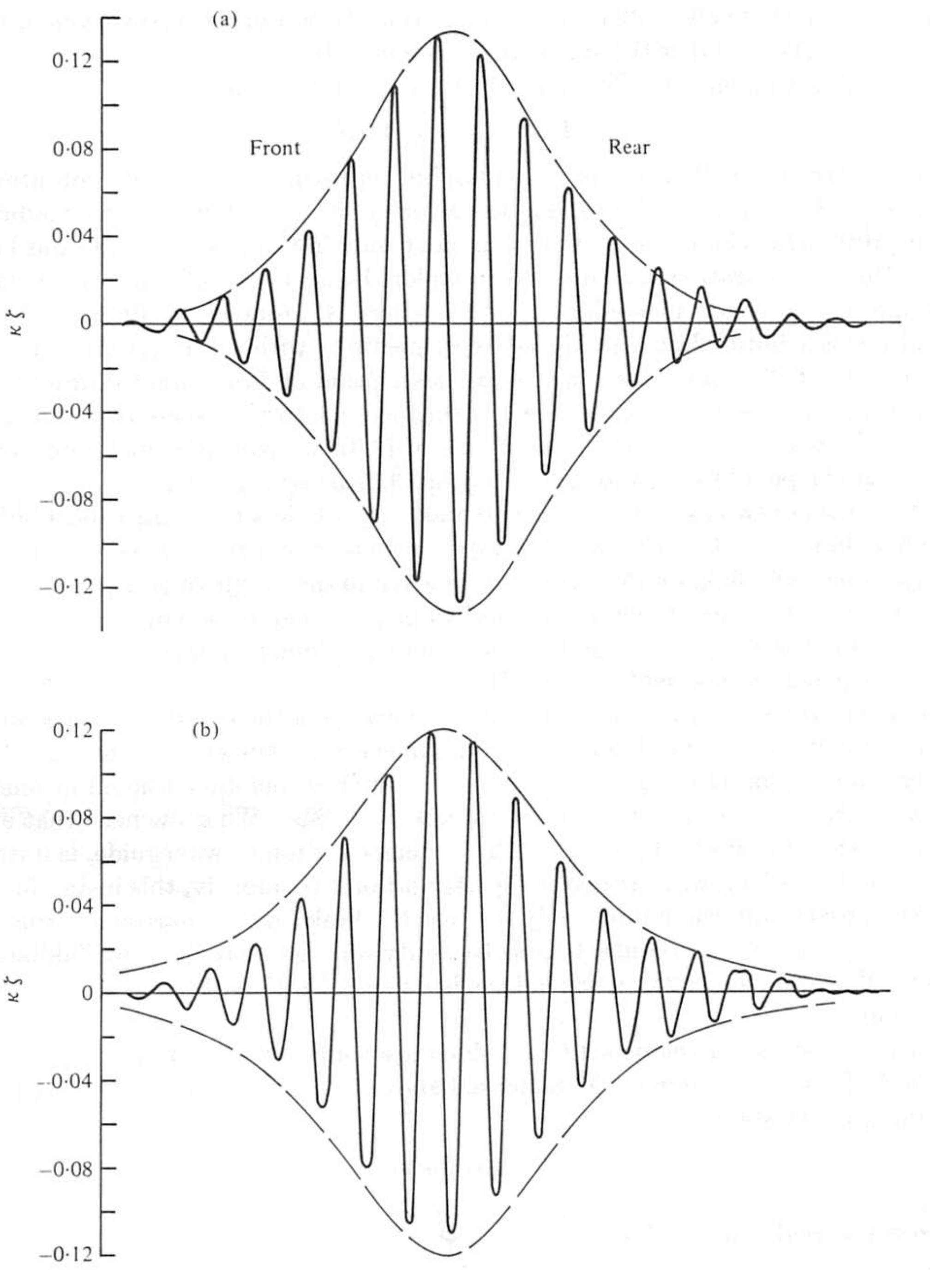

Figure 4: Measured surface displacement, showing evolution of envelope soliton at two downstream locations. Dimensions are $h = 1$ m, $kh = 4.0$, $\omega = 1$ Hz. The solid line is the measured history of surface displacement; the dashed line is the theoretical envelope shape. Data was taken at 6 m (top) and 30 m (bottom) downstream of the wavemaker.

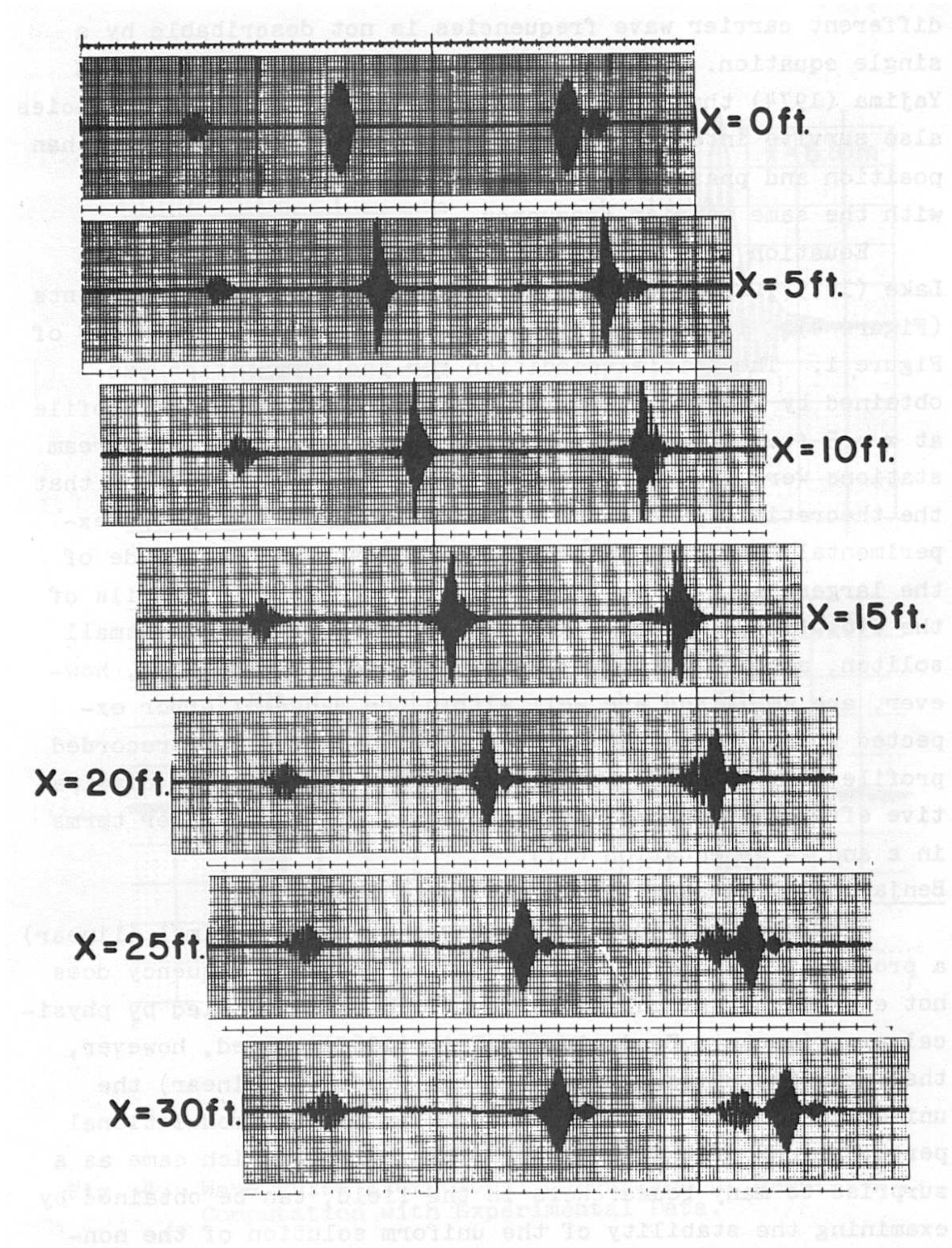

Figure 5: Wave pulse interaction: one wave pulse overtaking and passing through another wave pulse. Left-hand trace: first pulse alone, $\omega_0 = 1.5$ Hz, initial $(ka)_{max} \simeq 0.01$, six-cycle pulse. Center trace: second pulse alone, $\omega_0 = 3$ Hz, initial $(ka)_{max} \simeq 0.2$, 12-cycle pulse which disintegrates into two solitons. Right-hand traces: interaction of the two pulses. Adapted from [13].

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