Lecture17: Generalized Solitary Waves

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We have seen that solitary waves, either with a "pulse"-like profile or as the envelope of a wave packet, play a key role in nonlinear wave dynamics. However, there are physical situations when such KdV-type waves may not be genuinely localized. Instead they are accompanied by co-propagating small oscillations which spread out to infinity without decay (see Figure 1). These are *generalized solitary waves*. As we saw in Lecture 16, they may occur for water waves with surface tension for Bond numbers less than 1/3. It can be shown that they can also occur for interfacial waves when there is a free surface, and for all internal waves with mode numbers $n \geq 2$. The underlying reason for their existence is the presence of a resonance between a long wave with wave number $k \approx 0$ and a short wave with a finite wave number. When the amplitude of the central core is small compared to its length, $O(\epsilon^2)$, the amplitude of the oscillations is exponentially small, typically $O(\exp(-C/\epsilon))$ where C is a positive constant. Hence generalized solitary waves cannot usually be found by conventional asymptotic expansions, and need *exponential asymptotics*.

Consider the dispersion relation for internal waves, shown in Figure 2. Normally any mode numbers higher than 1 will resonate with other modes, so typically these waves do not persist, and we see only the mode 1 (soliton) waves. However, sometimes the first mode resonates with the surface mode, in which case waves of mode 2 or higher become generalized solitary waves, and only the mode 1 wave is a pure solitary wave.

Steady generalized solitary waves are necessarily symmetric. However, this means they cannot be realized physically as then the group velocity of the small oscillations is the same in both tails, which implies that energy sources and sinks are needed at infinity. In practice, these waves are generated asymetrically, with a core and small oscillations only on one side, determined by the group velocity (see Figure 3). Consequently, they are unsteady and slowly decay due to this radiation. In Figure 4 we present an acoustic visualisation of the streamlines generated by stratified flow past a sill, as reported in the experiments of Farmer & Smith [3]. This phenomenon was subsequently explained in terms of generalized solitary waves by Akylas & Grimshaw [1].

1 The Coupled KdV Equations

The technique we use to find the tail oscillations is based on extending the usual asymptotic expansion into the complex plane, and using Borel summation. It is similar to the techniques used by [6] and [7].

We begin by using a model system of two coupled KdV equations, which can be shown to describe the interaction between two weakly nonlinear long internal waves whose linear

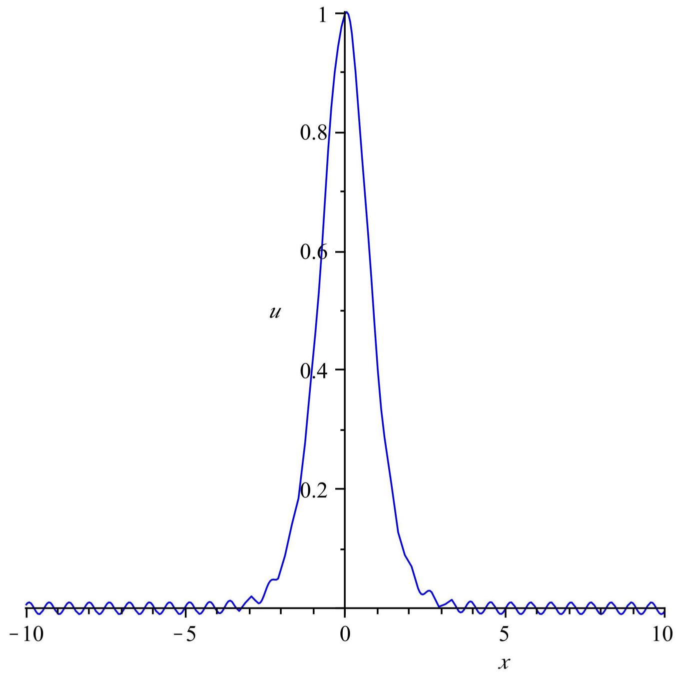

Figure 1: Schematic plot of a generalized solitary wave profile.

long wave speeds are nearly equal. The two coupled equations are

$$u_t + 6uu_x + u_{xxx} + \left(pv_{xx} + quv + \tfrac{1}{2}rv^2\right)_x = 0\,, \tag{1a}$$

$$v_t + \Delta v_x + 6vv_x + v_{xxx} + \lambda\left(pu_{xx} + ruv + \tfrac{1}{2}qu^2\right)_x = 0\,, \tag{1b}$$

where λ is the coupling parameter and Δ is the detuning parameter, proportional to the difference between the two linear long wave speeds, and p, q and r are real-valued constants. For stability we choose $\lambda > 0$, and we may also take $\Delta > 0$ without loss of generality. This system is Hamiltonian, and possesses conservation laws for the "mass" variables u and v, the "momentum" $\lambda u^2 + v^2$, and the Hamiltonian.

Let us first examine the linear spectrum for waves of wave number k and phase speed c for this system. Linearization of (1a) and (1b), followed by a search for solutions of the kind $e^{ik(x-ct)}$ yields

$$c = \tfrac{1}{2}\Delta - k^2 \pm \sqrt{\lambda p^2 k^4 + \tfrac{1}{4}\Delta^2}. \tag{2}$$

If we let the coupling parameter $\lambda \to 0$ these linear modes uncouple into a u-mode with spectrum $c = -k^2$ and a v-mode with spectrum $c = \Delta - k^2$. This situation persists for $\lambda > 0$, and there is a resonance between the long wave (u-mode) and a short wave (v-mode), with a resonant wavenumber $k_0 = \sqrt{\Delta/(1-\lambda p^2)}$ provided that $\lambda p^2 < 1$. A typical plot of these modes is presented in Figure 5.

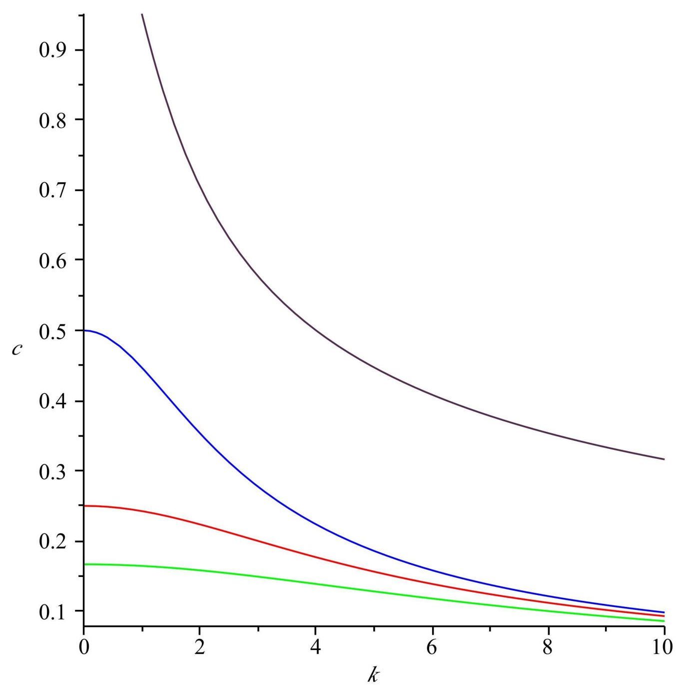

Figure 2: Plot of a schematic set of dispersion curves for internal waves: mode 1 (blue), mode 2 (red), mode 3 (green) and the surface mode (violet).

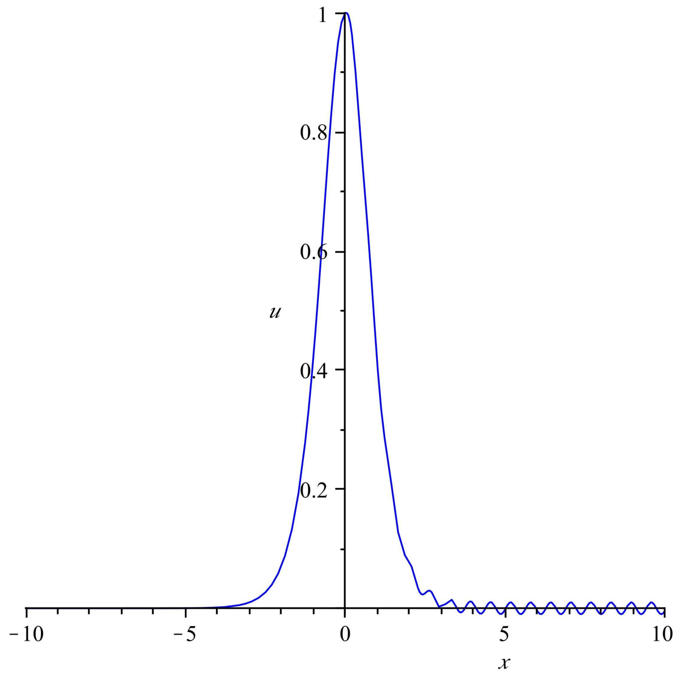

Figure 3: Schematic plot of an asymmetric generalized solitary wave profile.

We now seek nonlinear travelling wave solutions of the form

$$u = u(x - ct)\,, \quad v = v(x - ct)\,, \tag{3}$$

so that the coupled KdV system (1a, 1b) can be integrated once to become

$$-\,cu + 3u^2 + u_{xx} + pv_{xx} + quv + \tfrac{1}{2}rv^2 = 0\,, \tag{4a}$$

$$-\,cv + \Delta v + 3v^2 + v_{xx} + \lambda\left(pu_{xx} + ruv + \tfrac{1}{2}qu^2\right) = 0\,. \tag{4b}$$

Here the two constants of integration have been set to zero, which is achieved either by imposing solitary wave boundary conditions ($u, v \to 0$ as $|x| \to \infty$) or by translating u and v by constants. Equations (4a, 4b) form a fourth order ODE system. We shall show that they have symmetric generalized solitary wave solutions with co-propagating oscillatory tails of small amplitude. This amplitude will be found using either exponential asymptotics, or more directly by expanding in λ.

2 Exponential Asymptotics

A typical approach to these equations is to expand around $k = 0$ for the long (u-mode) wave. We introduce a small parameter $\epsilon \ll 1$, and seek an asymptotic expansion of the following form,

$$u_s(\epsilon x) = \sum_{n=1}^{\infty} \epsilon^{2n} u_n\,, \quad v_s(\epsilon x) = \sum_{n=1}^{\infty} \epsilon^{2n} v_n\,, \quad c = \sum_{n=1}^{\infty} \epsilon^{2n} c_n\,. \tag{5}$$

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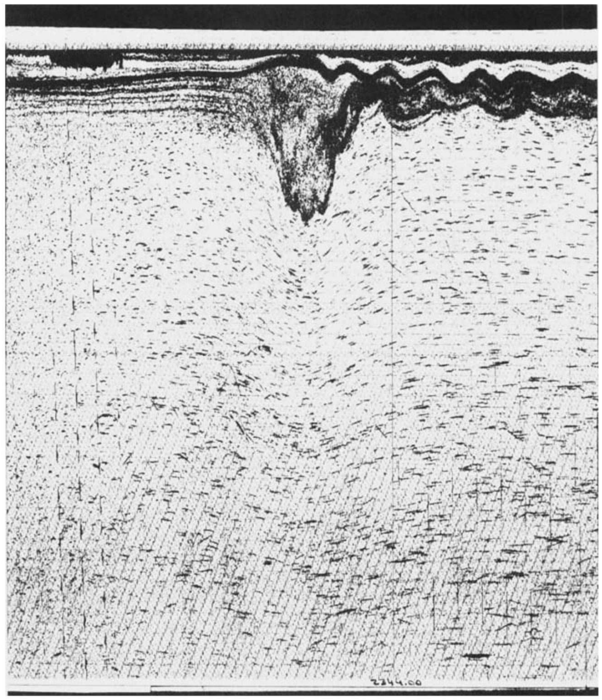

FIGURE 2. Acoustical image of internal-wave disturbances generated by stratified flow past a sill in the field experiments of Farmer & Smith (1980). The streamlines indicate that the main disturbance is a mode-2 solitary-like wave and is followed by a train of smaller-amplitude mode-1 short waves.

Recently, Turkington, Eydeland & Wang (1991), using a variational formulation of the governing equations, proposed a numerical technique for computing solitary-wave solutions in a stratified fluid, and presented several examples of mode-1 solitary waves; as expected, these waves are locally confined. In earlier related work, Tung, Chan & Kubota (1982) proved analytically and confirmed through numerical computations that large-amplitude locally confined mode-1 and mode-2 solitary waves are possible in a stratified fluid of finite depth, under the Boussinesq approximation. However, in discussing mode-2 solitary waves, they further assume that the density stratification is such that the Brunt–Väisälä frequency is symmetric about the fluid-layer centreline. This additional condition precludes the appearance of mode-1 oscillatory tails because waves of the first mode are symmetric while waves of the second mode are antisymmetric about the centreline. Nevertheless, mode-1 oscillations are still expected to develop at the tails of mode-3 solitary waves, which are also symmetric, but Tung *et al.* (1982) do not report calculations of solitary waves of mode-3 or higher.

Figure 4: An experimental observation from [1] of an asymmetric generalized solitary internal wave generated by stratified flow past a sill. The streamlines are visualised using acoustic imaging, and appear to show a mode 2 solitary wave followed by a train of smaller-amplitude mode-1 waves.

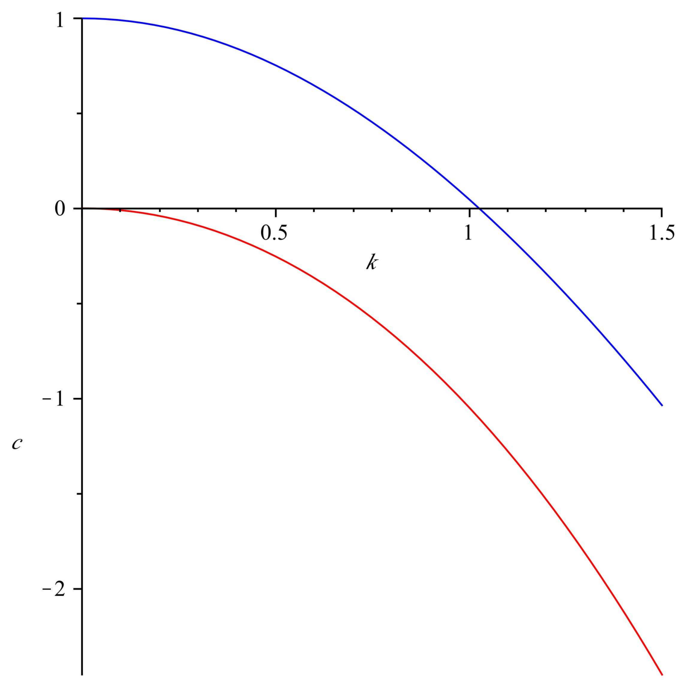

Figure 5: Plot of the linear phase speed c as a function of wave number k in the coupled KdV equations. Both the u-mode (red curve) and the v-mode (blue curve) are shown for the case $\Delta = 1$, $p = 0.5$, $\lambda = 0.2$.

Substituting this into (4a,4b) and solving order by order in ϵ yields

$$u_1 = 2\gamma^2 \mathrm{sech}^2(\epsilon\gamma x)\,, \quad v_1 = 0\,, \quad c_1 = 4\gamma^2\,, \tag{6a}$$

$$u_2 = \frac{\lambda}{\Delta}\left\{(20p^2 + q^2 - 8pq)c_1 u_1 - (q-6p)(q-10p)u_1^2\right\}\,, \tag{6b}$$

$$v_2 = -\frac{\lambda}{\Delta}\left\{pc_1 u_1 + \tfrac{1}{2}(q-6p)u_1^2\right\}\,, \tag{6c}$$

$$c_2 = -\frac{\lambda}{\Delta}p^2 c_1^2\,. \tag{6d}$$

The expansion can be continued to all orders in ϵ^2 without any oscillatory tail being detected. This is because the size of the tail depends exponentially on ϵ, and so it decays faster as $\epsilon \to 0$ than any power of ϵ.

To find the tail oscillations, we observe that u_n, v_n are singular in the complex plane at $x = (2m+1)i\pi/2\epsilon\gamma$, $m \in \mathbb{Z}$. This motivates a closer examination via the change of variable

$$x = \frac{i\pi}{2\epsilon\gamma} + z\,, \tag{7}$$

which allows us to consider the region of the complex plane close to the first singularity. Then as $\epsilon z \to 0$, $\mathrm{sech}^2(\epsilon\gamma x) \sim -1/\epsilon^2\gamma^2 z^2$, and so, substituting back into our asymptotic expansion (6a)–(6d),

$$u_s \sim -\frac{2}{z^2} - \frac{\lambda}{2\Delta z^4}(q-6p)(q-10p) + \cdots + O(\epsilon^2)\,, \tag{8a}$$

$$v_s \sim -\frac{2\lambda}{\Delta z^4}(q-6p) + \cdots + O(\epsilon^2)\,. \tag{8b}$$

Next we consider the inner problem, in which we seek solutions of (4a, 4b) in the form $u = u(z), v = v(z)$, and for which the expressions (8a, 8b) form an outer boundary condition. The outcome is just the same system (4a, 4b) with x replaced by z,

$$-\,cu + 3u^2 + u_{zz} + pv_{zz} + quv + \tfrac{1}{2}rv^2 = 0\,, \tag{9a}$$

$$-\,cv + \Delta v + 3v^2 + v_{zz} + \lambda\left(pu_{zz} + ruv + \tfrac{1}{2}qu^2\right) = 0\,. \tag{9b}$$

Note that $c = O(\epsilon^2)$ from (6a) and can be omitted at the leading order. We proceed by applying a Laplace transform

$$[u, v] = \int_\Gamma e^{-zs}[U(s), V(s)]\,ds\,, \tag{10}$$

where the contour Γ runs from 0 to ∞ in the half-plane $\mathrm{Re}\{sz\} > 0$. We then seek a power series solution

$$[U(s), V(s)] = \sum_{n=1}^{\infty}[a_n, b_n]s^{2n-1}\,, \tag{11}$$

where $a_1 = -2$, $b_1 = 0$, $a_2 = -\lambda(q-6p)(q-10p)/12\Delta$, $b_2 = -\lambda(q-6p)/3\Delta$ from (8a, 8b). In general, substitution of (11) into the Laplace transform (10) generates the asymptotic series

$$[u, v] \sim \sum_{n=1}^{\infty}[\alpha_n, \beta_n]z^{-2n}\,, \quad [\alpha_n, \beta_n] = (2n-1)![a_n, b_n]\,. \tag{12}$$

This agrees with the asymptotic series (8a, 8b), and in effect the Laplace transform is a Borel summation of the asymptotic series.

Substitution of the Laplace transform (10) and the series (11) into the differential equation system (9a, 9b) yields a recurrence relation for $[a_n, b_n]$. Setting $\Delta[A_n, B_n] = (-k_0^2)^n[a_n, b_n]$, we find that

$$\frac{(n+1)(2n+5)}{(n-1)(2n-1)}A_{n-1} + \left(p - \frac{q}{(n-1)(2n-1)}\right) B_{n-1} = F_n \,, \tag{13}$$

$$\left(1 - \lambda p^2\right) B_n - B_{n-1} - \lambda p A_{n-1} + \lambda \frac{rB_{n-1} + qA_{n-1}}{(n-1)(2n-1)} = G_n \,, \tag{14}$$

where F_n and G_n are quadratic convolution expressions in $A_2, \ldots, A_{n-2}$ and $B_2, \ldots, B_{n-2}$. As $n \to \infty$, these nonlinear terms can be neglected, and we find that

$$[A_n, B_n] \to [-p, 1]K \quad \text{as} \quad n \to \infty \,, \tag{15}$$

where K is a constant whose value depends on λ, p, q, r. It follows that the series (11) converges for $|s| < k_0$, $k_0^2 = \Delta/(1 - \lambda p^2)$. The result (15) shows that as $|s| \to k_0$ there is a pole singularity given by

$$[U(s), V(s)] \approx \Delta \frac{[p, -1]K}{2(s - ik_0)} \,. \tag{16}$$

We have now established that the solution in the z-variable is given by (10) where $[U(s), V(s)]$ has a pole singularity at $s = ik_0$, at the complex conjugate point $s = -ik_0$, and at all of their harmonics $s = \pm imk_0$, $m \in \mathbb{N}$. Hence the contour Γ should be chosen to avoid the imaginary s-axis, and to be explicit we choose it to lie in $\mathrm{Re}\{s\} > 0$. However, we seek a symmetric solution, which in the z-variable requires that $\mathrm{Im}\{u, v\} = 0$ when $\mathrm{Re}\{z\} = 0$. The presence of the pole prevents (10) from satisfying this condition, and so we must correct it by adding a subdominant term

$$[u, v] = \int_\Gamma e^{-zs}[U(s), V(s)]\, ds + \frac{ib}{2}[p, -1] \exp(-ik_0 z + i\delta). \tag{17}$$

Here b, δ are real constants, and note that $|\exp(-ik_0 z)|$ is smaller than any power of $|z|^{-1}$ as $z \to \infty$ in $\mathrm{Re}\{z\} > 0, \mathrm{Im}\{z\} < 0$, recalling that $x = (i\pi/2\epsilon\gamma) + z$. The symmetry condition is now applied by bringing the contour Γ onto $\mathrm{Re}\{s\} = 0$ and deforming around the pole at $s = ik_0$. The outcome is

$$b \cos\delta = \pi K \,, \tag{18}$$

which we substitute into (17). The final step is to bring this solution back to the real axis, using $x = (i\pi/2\epsilon\gamma) + z$. Taking account of the corresponding singularity at $s = -ik_0$, we finally arrive at

$$[u, v] \sim [u_s, v_s] + b\Delta[-p, -1] \exp(-\pi k_0/2\epsilon\gamma) \sin(k_0|x| - \delta) \,. \tag{19}$$

This is a two-parameter family in γ and δ, where $0 < \delta < \pi/2$. The minimum tail amplitude occurs at $\delta = 0$, whilst the amplitude tends to infinity as $\delta \to \pi/2$. Note that the constant argument of the exponential is determined by the location of the singularity, but we require the exponential asymptotics to find the amplitude.

3 Embedded solitons

The constant K is determined by the recurrence relations (13, 14). It is a function of the system parameters λ, p, q, r and in general is found numerically. However, from our straightforward asymptotic solution (6a, 6b) we know that $K = 0$ for $q = 6p$, and in general we may find many parameter combinations where $K = 0$. In particular,

$$K \sim \frac{\lambda(6p - q)}{3\Delta} \quad \text{as} \quad \lambda \to 0\,. \tag{20}$$

These special values imply that the solitary wave decays to zero at infinity, even though its speed lies inside the linear spectrum, at least in this asymptotic limit. These are called embedded solitons. They are usually not stable, but are then *metastable*, or are said to exhibit *semi-stability*, in that they are unstable to small, but not infinitesimal, perturbations. Nevertheless, they are found to be useful in several applications, such as nonlinear optics and solid state physics. For water waves with surface tension, generalized solitary waves exist for Bond numbers $0 < B < 1/3$, although numerical simulations suggest that there are no embedded solitons.

4 One-sided generalized solitary waves

These symmetric solitary waves cannot be realized in practice, since they require an energy source and sink at infinity. Instead, they are replaced by solitary waves with radiating tails on one side only, determined by the group velocity. That is, in $x > 0$ for $c_g > c$, or in $x < 0$ for $c_g < c$, where c_g is the group velocity at the resonant wavenumber. For the present case, the linear dispersion relation is (2) and so for the relevant u-mode, $c_g = \Delta - 3k^2 < c = \Delta - k^2$. Hence there are no oscillations in $x > 0$, but they will appear in $x < 0$.

Thus in $x > 0$, or more generally in $\mathrm{Re}\{z\} > 0$, the solution is completely defined by the Laplace transform integral (10), with the contour Γ lying in $\mathrm{Re}\{s\} > 0$. Then for $x < 0$, or $\mathrm{Re}\{z\} < 0$, the contour Γ must be moved to $\mathrm{Re}\{z\} < 0$ across the axis $\mathrm{Re}\{s\} = 0$. In this process the solution collects a contribution from the pole at $s = ik_0$, which generates the tail oscillation. The final outcome is that (19) is replaced by

$$[u, v] \sim [u_s, v_s] - H(-x) 2\pi K \Delta [-p, -1] \exp(-\pi k_0/2\epsilon\gamma) \sin(k_0 x) \tag{21}$$

where $H(\cdot)$ is the Heaviside function. In effect the phase shift $\delta = 0$, there are no oscillations in $x > 0$, and the amplitude in $x < 0$ is exactly twice the amplitude of the symmetric solution.

5 Weak Coupling

Let us now return to the coupled travelling wave equations (4a, 4b). Supposing that the coupling parameter is very small, $0 < \lambda \ll 1$, we may expand asymptotically in λ as follows,

$$[u, v] \sim \sum_{n=0}^{\infty} \lambda^n [u_n, v_n]\,, \quad c \sim \sum_{n=0}^{\infty} \lambda^n c_n\,. \tag{22}$$

Substituting this expansion into (4a 4b), we find that the leading order solution is

$$u_0 = 2\beta^2 \mathrm{sech}^2(\beta x)\,, \quad v_0 = 0\,, \quad c_0 = 4\beta^2\,. \tag{23}$$

This leading term is a u-mode solitary wave. A comparison with the previous expansion (5) suggests that $\beta = \epsilon\gamma$, but now the amplitude can be order unity. At the next order

$$-c_0 u_1 + 6u_0 u_1 + u_{1xx} + p v_{1xx} + q u_0 v_1 - c_1 u_0 = 0\,, \tag{24a}$$

$$(\Delta - c_0) v_1 + v_{1xx} + p u_{0xx} + \tfrac{1}{2} q u_0^2 = 0\,, \tag{24b}$$

We use the leading order solution for u (23) to rewrite (24b) as

$$(\Delta - c_0) v_1 + v_{1xx} = f(x) = -p c_0 u_0 + \tfrac{1}{2}(6p - q) u_0^2\,. \tag{25}$$

Note that in the limit $\lambda \to 0$, the resonant wavenumber is $k_0 \approx (\Delta - c_0)^{1/2}$, which takes account of the finite speed of the wave. We must now take $c_0 < \Delta$ to get tail oscillations, whilst for $c_0 > \Delta$ the expansion yields a genuine solitary wave. The general solution of (25) is

$$v_1 = A \sin k_0 x + B \cos k_0 x + \frac{1}{2k_0} \int_{-\infty}^{\infty} f(x') \sin(k_0 |x - x'|) dx'\,. \tag{26}$$

To determine the constants A, B we impose a symmetry condition on v_1, so that $A = 0$, and then

$$v_1 \sim b_1 \sin(k_0 |x| - \delta) \quad \text{as} \quad |x| \to \infty\,, \tag{27}$$

$$b_1 \cos\delta = L = \frac{1}{2k_0} \int_{-\infty}^{\infty} f(x) \cos(k_0 x) dx\,. \tag{28}$$

With v_1 known, we take the limit $|x| \to \infty$ in (24a) to find

$$u_1 \sim -p \frac{(\Delta - c_0)}{\Delta} b_1 \sin(k_0 |x| - \delta)\,, \quad \text{as} \quad |x| \to \infty\,, \tag{29}$$

Substituting (25) into (28), we find that

$$L = -\frac{\beta^2}{6k_0} \left\{ k_0^2 (q - 6p) + 4\beta^2 q \right\} \int_{-\infty}^{\infty} \mathrm{sech}^2(\beta x) \cos(k_0 x) dx\,. \tag{30}$$

Then, as $\beta = \epsilon\gamma \to 0$, this reduces to

$$L \sim \frac{\pi k_0^2}{3} (6p - q) \exp(-\pi k_0 / 2\epsilon\gamma)\,, \tag{31}$$

which agrees with the previous result (20) from the exponential asymptotics, since $L = \pi K$. The one-sided solutions are obtained by setting $\delta = 0$, and replacing b_1 in (27, 29) by either 0 for $x > 0$ or $2b_1$ for $x < 0$.

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