Lecture 18: Wave-Mean Flow Interaction, Part I

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1 Introduction

Nonlinearity in water waves can lead to wave breaking. We can observe easily that waves break as they come to a beach. The waves on a current also may break. In this lecture we derive modulation equations of water waves using Whitham's averaged Lagrangian method. Then we consider the interaction of nonlinear water waves with currents and slopping bottom topography (e.g. waves on a beach).

2 Water waves in the linear approximation

In the linear approximation, the surface elevation ζ for sinusoidal unidirectional waves is

$$\zeta(x,t) = a\cos\theta, \quad \theta = kx - \omega t + \alpha, \tag{1}$$

for waves of amplitude a, wavenumber k (> 0), and frequency ω. Here α is an arbitrary constant ensemble parameter. When there is no mean current the dispersion relation is

$$\omega^2 = gk\tanh(kH), \tag{2}$$

where g is the acceleration due to gravity and H is the mean water depth.

If there is a constant horizontal mean current U, in the frame moving with the current ($x' = x - Ut$), the dispersion relation remains similar to (2). Then, θ can also be written as

$$\theta = kx - \omega t + \alpha = k(x' + Ut) - \omega t + \alpha = kx' - (\omega - kU)t + \alpha, \tag{3}$$

so that the dispersion relation of water waves on a horizontal mean current, in the *rest* frame is

$$\omega = Uk + \omega^*. \tag{4}$$

where ω^* is the intrinsic frequency (i.e. $\omega^* = \pm[gk\tanh(kH)]^{1/2}$), which has two branches. The total frequency ω is thus decomposed into the Doppler shift Uk and the intrinsic frequency ω^*.

3 Modulation equations of water waves

Now suppose that the amplitude, wave number, frequency, mean current and mean depth vary slowly relative to the wave field. Then (1) is replaced by the Fourier series expansions as

$$\zeta(x,t) \sim a(x,t)\cos\theta + a_2\cos 2\theta + O(a^3), \tag{5}$$

$$\theta = \phi(x,t) + \alpha, \quad k = \frac{\partial\phi}{\partial x}, \quad \omega = -\frac{\partial\phi}{\partial t}. \tag{6}$$

Here ϕ is the phase, and the ensemble parameter α is constant, and the coefficient $a_2 \sim O(a^2)$ depends on ω^*, k, U, H. It is convenient to introduce a velocity potential Ψ defined as

$$\Psi = Ux - Bt + \Phi(\theta, z), \tag{7}$$

where Φ is the wave component of Ψ, and B is related to the mean height of the waves. Now, Φ is expanded as Fourier series in the form

$$\Phi(\theta, z) = A_1\cosh(kz)\sin\theta + A_2\cosh(2kz)\sin 2\theta + O(a^3), \tag{8}$$

where $A_1 \sim O(a)$ and $A_2 \sim O(a^2)$. This is because of the kinematic boundary condition at free surface $\frac{\partial\Psi}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{\partial\Psi}{\partial x}\frac{\partial\zeta}{\partial x}$. From (6), the equation for conservation of waves is,

$$\frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0. \tag{9}$$

The issue is to determine how the amplitude, wavenumber, frequency, mean current and mean depth vary (slowly) in space and time. The mean current $U(x,t)$ and depth $H(x,t)$ can be decomposed into background components $u(x,t)$, $h(x)$ and a wave-induced $O(a^2)$ component.

The modulation equations for the wave amplitude, wavenumber, frequency, mean current and mean depth are found using Whitham's averaged Lagrangian method. The Lagrangian of the water wave system is (Whitham, 1974, chapter 13.2)

$$L = -\int_{-h}^{\zeta}\left\{\frac{\partial\Psi}{\partial t} + \frac{1}{2}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\Psi}{\partial z}\right)^2 + gz\right\}\mathrm{d}z. \tag{10}$$

Substituting (7) into (10), we obtain

$$\begin{aligned} L &= \int_{-h}^{\zeta}\left[B + \omega\frac{\partial\Phi}{\partial\theta} - \frac{1}{2}\left(U + k\frac{\partial\Phi}{\partial\theta}\right)^2 - \frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2 - gz\right]\mathrm{d}z, \\ &= \left(B - \frac{1}{2}U^2\right)H - \frac{1}{2}gH^2 + gHh \\ &\quad + (\omega - Uk)\int_{-h}^{\zeta}\frac{\partial\Phi}{\partial\theta}\mathrm{d}z - \int_{-h}^{\zeta}\frac{1}{2}\left[k^2\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\mathrm{d}z. \end{aligned} \tag{11}$$

Note that $\zeta + h = H$ and $\zeta^2 - h^2 = (\zeta + h)^2 - 2(\zeta + h)h$.

Averaging the Lagrangian (11) in α, we obtain

$$\overline{L} = \frac{1}{2\pi}\int_0^{2\pi} L\mathrm{d}\alpha = \overline{L}^{(\mathrm{m})}(U,B,H,h) + \overline{L}^{(\mathrm{w})}(E^*,\omega^*,k,H), \tag{12}$$

$$E^* = \frac{ga^2}{2}, \quad k = \frac{\partial\phi}{\partial x}, \quad \omega = -\frac{\partial\phi}{\partial t}, \quad U = \frac{\partial\Psi}{\partial x}, \quad B = -\frac{\partial\Psi}{\partial t}. \tag{13}$$

The functions $\overline{L}^{(\mathrm{m})}(U,B,H,h)$ and $\overline{L}^{(\mathrm{w})}(E^*,\omega^*,k,H)$ are mean and wave components of averaged Lagrangian respectively.

$$\text{Mean} \quad : \quad \overline{L}^{(\mathrm{m})} = \left(B - \frac{U^2}{2}\right)H - \frac{gH^2}{2} + gHh, \tag{14}$$

$$\text{Wave} \quad : \quad \overline{L}^{(\mathrm{w})} = \frac{DE^*}{2} + \frac{D_2k^2E^{*2}}{2g} + O(E^{*3}), \tag{15}$$

where

$$D = \frac{\omega^{*2}}{gkT} - 1, \quad D_2 = -\frac{9T^4 - 10T^2 + 9}{8T^4}, \quad T = \tanh(kH), \quad \omega^* = \omega - Uk. \tag{16}$$

These expressions are derived first by finding Ψ and thus L to get $\bar{L}$. The coefficients A_1, A_2, a_2 in (7) are obtained by solving the variational equations

$$\frac{\partial\overline{L}}{\partial A_1} = 0, \quad \frac{\partial\overline{L}}{\partial A_2} = 0, \quad \frac{\partial\overline{L}}{\partial a_2} = 0, \tag{17}$$

for A_1, A_2, a_2. The resulting Φ is then used to find Ψ (8), which is then re-substituted into L (11) and then $\overline{L}$. See Whitham (1974, chapter 16.6) and Whitham (1967) for details.

To obtain the modulation equations, the averaged variational principle

$$\delta\int\int \overline{L}dxdt = 0, \tag{18}$$

is used for variations in $\delta E^*, \delta\phi, \delta\psi, \delta H$, and we have

$$\delta E^* \quad : \quad \frac{\partial\overline{L}}{\partial E^*} = 0, \tag{19}$$

$$\delta\phi \quad : \quad \frac{\partial}{\partial t}\left(\frac{\partial\overline{L}}{\partial\omega}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\overline{L}}{\partial k}\right) = 0, \quad \frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0, \tag{20}$$

$$\delta\Psi \quad : \quad \frac{\partial}{\partial t}\left(\frac{\partial\overline{L}}{\partial B}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\overline{L}}{\partial U}\right) = 0, \quad \frac{\partial U}{\partial t} + \frac{\partial B}{\partial x} = 0, \tag{21}$$

$$\delta H \quad : \quad \frac{\partial\overline{L}}{\partial H} = 0. \tag{22}$$

From (19), the dispersion relation is obtained as

$$\frac{D}{2} + \frac{k^2D_2E^*}{g} + O(E^{*2}) = 0. \tag{23}$$

From (21) we obtain

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[UH + k\left(E\frac{(\omega - Uk)}{gkT}\right)\right] = \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(HV) = 0, \tag{24}$$

where

$$V = U + \frac{kA}{H}, \quad A = \frac{\partial \overline{L}}{\partial \omega} = E\frac{(\omega - Uk)}{gkT}. \tag{25}$$

From (22) we obtain

$$B = \frac{1}{2}U^2 + gH - gh + \frac{1}{2}\left(\frac{1 - T^2}{T}\right)kE^* + O(E^{*2}). \tag{26}$$

Using (25) and (26), we obtain

$$\frac{\partial}{\partial t}(HV) + \frac{\partial}{\partial x}(HV^2) + \frac{\partial}{\partial x}\left(\frac{gH^2}{2}\right) + \frac{\partial S}{\partial x} = gH\frac{\partial h}{\partial x}, \tag{27}$$

where

$$S = k(F - VA) + \overline{L}^{(\mathrm{w})} - H\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial H}, \quad F = -\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial k}. \tag{28}$$

Equation (20) can be written as

$$\frac{\partial A}{\partial t} + \frac{\partial F}{\partial x} = 0. \tag{29}$$

Now, (19), (29), (24), and (27) are the dispersion relation, the wave action equation, the mean flow and mean momentum equations of the modulation equations respectively. To these we add equation (9) for conservation of waves.

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{30}$$

These equations are fully nonlinear. A is the wave action density and F is the wave action flux. In the linearized approximation ($H \approx h$, $U \approx u$) the dispersion relation (23) becomes

$$D(\omega^*, k, h) = 0, \quad \omega = \omega^* + ku, \quad \omega^{*2} = gk\tanh(kh) \tag{31}$$

$$A = \frac{\partial D}{\partial \omega^*}\frac{E^*}{2} = \frac{E^*}{\omega^*}, \quad F = c_g A = (c_g^* + u)A, \tag{32}$$

where $c_g^* = \frac{\partial \omega^*}{\partial k}$ is the intrinsic group velocity. S is the radiation stress, which in the linearized approximation reduces to

$$S = \left(kc_g^* + h\frac{\partial \omega^*}{\partial h}\right)A = \left(2kc_g^* - \frac{\omega^*}{2}\right)A, \tag{33}$$

since for water waves,

$$h\frac{\partial \omega^*}{\partial h} = kc_g^* - \frac{\omega^*}{2}. \tag{34}$$

The equation for conservation of waves (30) becomes

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} = -k\frac{\partial u}{\partial x} - \frac{\partial \omega^*}{\partial h}\frac{\partial h}{\partial x}, \quad \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} = k\frac{\partial u}{\partial t}. \tag{35}$$

Note that for steady backgrounds the frequency is conserved.

4 Waves on a current

Now we consider a unidirectional steady current $u = u(x)$, with constant depth h.

4.1 Linear approximation

In the linearized approximation ($H \approx h$, $U \approx u$), equation (30) becomes

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \quad \omega = uk + \omega^*, \quad \omega^{*2} = gk \tanh(kh). \tag{36}$$

The steady solution is $\omega = \omega_0$ (constant), with $k = k(x)$. The wave amplitude is obtained from the wave action equation (29), which reduces to

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(c_g A) = 0, \quad c_g = u + c_g^*, \quad A = \frac{E^*}{\omega^*}. \tag{37}$$

The steady solution has constant wave action flux F_0,

$$2c_g A = \frac{c_g c^* a^2}{\tanh(kh)} = 2F_0, \quad c^* = \frac{\omega^*}{k}. \tag{38}$$

For simplicity, we now make the deep-water approximation $kh \to \infty$, so that $\omega^{*2} = gk$, $c_g^* = c^*/2$. Suppose that $u(x = 0) = 0$ and the intrinsic phase speed is $c^* = c_0 > 0$ at $x = 0$. Then the steady solution of (36) is obtained as follows. Since $\omega = \omega_0$,

$$u(x)k(x) + \sqrt{gk(x)} - \sqrt{gk(0)} = 0. \tag{39}$$

Dividing (39) by $k(x)$, we obtain

$$u(x) + c^*(x) - \frac{c^{*2}(x)}{c_0} = 0. \tag{40}$$

Here we used $c^*(x) = \sqrt{g/k(x)}$, $c_0 = c^*(0) = \sqrt{g/k(0)}$. Then the solution of (40) is

$$c^*(x) = \frac{c_0}{2} \pm \left\{ c_0 u(x) + \frac{c_0^2}{4} \right\}^{1/2}. \tag{41}$$

The condition at $x = 0$ means we choose the plus sign. Note that the group velocity is

$$c_g(x) = u(x) + \frac{c^*}{2} = u(x) + \frac{c_0}{4} \pm \frac{1}{2} \left\{ c_0 u(x) + \frac{c_0^2}{4} \right\}^{1/2}. \tag{42}$$

Thus, for an advancing current $u(x) > 0$, $x > 0$, we must choose only the plus sign, and so $c^*(x)$, $c_g(x)$ both increase as $u(x)$ increases, while then $k(x) = g/c^{*2}$ decreases. Since $c_g c^* a^2 = 2F_0$, the wave amplitude decreases.

For an opposing current $u(x) < 0$, $x > 0$, there is a stopping velocity at $x = x_c$, $u(x_c) = -c_0/4$, and the waves cannot penetrate past this point, since $c_g(x_c) = 0$. Instead the waves reflect, with the minus sign in (41, 42). Both $c^*(x)$, $c_g(x)$ decrease as $|u(x)|$ increases, while $k(x)$ increases. Since $c_g c^* a^2 = 2F_0 = c_0^2 a_0^2$, the wave amplitude increases from the initial value a_0, and $a^2 \to \infty$ as $x \to x_c$. Of course, this result is outside the linear approximation, and in practice the waves will break at $x_b < x = x_c$. Here we use an experimental breaking criterion, $ak(x_b) = 0.44$ (Miche, 1944); note that x_b depends on a_0 and c_0. Using (41, 42) and $c_g c^* a^2 = 2F_0 = c_0^2 a_0^2$, the wave steepness ak can be calculated as follows,

$$ak = a_0 k_0 (G_1 G_2)^{-1/2} G_1^{-2}, \tag{43}$$

where

$$G_1 = \frac{1}{2} + \left(\frac{u}{c_0} + \frac{1}{4}\right)^{1/2}, \tag{44}$$

$$G_2 = \frac{u}{c_0} + \frac{1}{2} G_1. \tag{45}$$

The relation between u/c_0 and ak (i.e. equation (43)) is shown as Figure 1.

This rather simple theory has applications to the formation of giant (rogue, freak) waves in the ocean, for example on the Agulhas current. There are also applications to the modulation of water waves by an underlying internal solitary wave, whose surface current is $u(x) = u_0 \operatorname{sech}^2(Kx)$ say (Fig. 2). To explore these further, we take a wave packet solution of the wave action equation (37)

$$c_g A = c_g c^* a^2 = c_0^2 a_0^2 b^2 (t - \tau), \quad \tau = \int_0^x \frac{dx}{c_g}. \tag{46}$$

Here $a_0 b(t)$ is the wave amplitude at $x = 0$, and we assume that the shape function $b(t)$ is localized (e.g. Gaussian), varying from 0 to a maximum of 1 at $t = 0$. Then the waves break throughout the zone, $x_b < x < x_c$, over a time interval determined by the width of the packet.

4.2 Nonlinear effects

In deep water, the wave-induced components of U, H are negligible and so the Lagrangian (12) becomes just (15) given now by

$$\overline{L}^{(\mathrm{w})} = \left(\frac{\omega^*}{gk} - 1\right) \frac{E^*}{2} - \frac{k^2 E^{*2}}{2g} + O(E^{*3}), \tag{47}$$

where now $\omega^* = \omega - ku(x)$. The nonlinear dispersion relation (23) becomes

$$\omega^{*2} = gk + 2k^3 E^* + O(E^{*2}). \tag{48}$$

Conservation of wave action (29) and conservation of waves (30) again yield for a steady solution ($\frac{\partial}{\partial t} = 0$)

$$F = -\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial k} = F_0, \quad \omega_0 = \omega^* + u(x)k, \tag{49}$$

where F_0, ω_0 are constants. When combined with (48) these yield two coupled equations for k, E^* in terms of $u(x)$. Now the dispersion relation (48) depends on the amplitude,

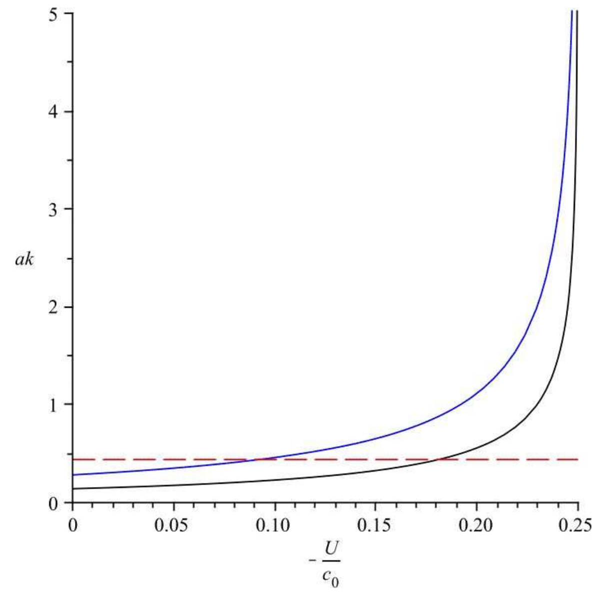

Figure 1: Wave steepness ak versus u/c_0; $a_0k_0 = 0.1, 0.2$ (black, blue), where a_0k_0 are wave steepness at $x = 0$. Wave breaking criterion $ak = 0.44$ (red dash), yields breaking for $|u|/c_0 > 0.18, 0.092$.

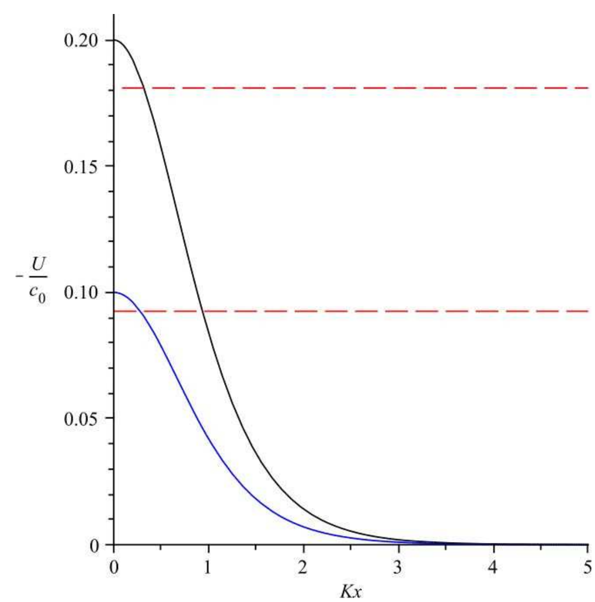

Figure 2: Breaking waves on the internal wave current $u = u_0\text{sech}^2(Kx)$, for $u_0/c_0 = -0.2, -0.1$ (black, blue), where the red lines give the breaking zones for $a_0k_0 = 0.1, 0.2$ (upper, lower). This shows that the waves of $a_0k_0 = 0.1$ (0.2) will break when they are on $|Kx| < 0.33$ (0.94) if the waves are on internal wave current indicated by black line.

$\omega^* = \omega^*(k, E^*)$ as well as the wavenumber. Conservation of wave action flux becomes

$$WA = F_0, \quad W = -\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial k} \Bigg/ \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial \omega} = u(x) + \frac{\omega^*}{2k} + k^2 A, \tag{50}$$

$$A = \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial \omega} = \frac{E^*}{\omega^*}\left(1 + \frac{2k^2 E^*}{g}\right). \tag{51}$$

These are combined with (48) and (49),

$$\omega^{*2} = gk + 2k^3 \omega^* A, \quad \omega_0 = \omega^* + u(x)k, \tag{52}$$

to yield two equations for k, A in terms of $u(x)$. Note that for an opposing current $u(x) < 0$ ($x > 0$) there is now no stopping velocity, as $W \to 0$, $A \to \infty$ is not allowed.

The equation of wave steepness ak in terms of $u(x)/c_0$ is obtained by the same method used to derive (43), but is more complicated. For convenience, we define

$$s(ak) = \omega^* k^2 A, \quad s_0 = s(a_0 k_0), \tag{53}$$

and

$$\hat{s} = \frac{g + 2s_0}{g + 2s}. \tag{54}$$

Then the first equation of (52) is written as

$$\omega^* = k(g + 2s), \tag{55}$$

so A and W can be written as follows,

$$\begin{aligned} A &= \frac{E^*}{\omega^*}\left(1 + \frac{2k^2 E^*}{g}\right), \\ &= \frac{c^* a^2 g}{2(g+2s)}(1 + a^2 k^2), \end{aligned} \tag{56}$$

$$\begin{aligned} W &= u + \frac{\omega^*}{2k} + k^2 A, \\ &= u + \frac{c^*}{2} + \frac{c^* a^2 k^2 g}{2(g+2s)}(1 + a^2 k^2), \\ &= u + c^* \left[\frac{1}{2} + \frac{a^2 k^2 g}{2(g+2s)}(1 + a^2 k^2)\right], \end{aligned} \tag{57}$$

where $c^* = \omega^*/k = \sqrt{g + 2s}/k$. Because $u(x = 0) = 0$, A_0 and W_0 are written as

$$A_0 = c_0 a_0^2 \frac{g}{2(g + 2s_0)}(1 + a_0^2 k_0^2), \tag{58}$$

$$W_0 = c_0 \left[\frac{1}{2} + \frac{a_0^2 k_0^2 g}{2(g + 2s_0)}(1 + a_0^2 k_0^2)\right]. \tag{59}$$

From the second equation of (52), we obtain

$$\frac{k_0}{k} = \frac{c^*}{c_0} + \frac{u}{c_0}. \tag{60}$$

Using (54), k_0/k can be written as

$$\frac{k_0}{k} = \hat{s}\left(\frac{c^*}{c_0}\right)^2. \tag{61}$$

From (60) and (61), we obtain the quadratic equation for c^*/c_0 as

$$\hat{s}\left(\frac{c^*}{c_0}\right)^2 - \left(\frac{c^*}{c_0}\right) - \frac{u}{c_0} = 0, \tag{62}$$

and its solutions are

$$\frac{c^*}{c_0} = \frac{-1 \pm \sqrt{1 + 4\hat{s}(u_0/c_0)}}{2\hat{s}}. \tag{63}$$

Conservation of wave action flux can be written as

$$WA = W_0 A_0. \tag{64}$$

Multiplying (64) by $(60)^2$, we obtain

$$WA\left(\frac{c^*}{c_0} + \frac{u}{c_0}\right)^2 = W_0 A_0 \left(\frac{k_0}{k}\right)^2. \tag{65}$$

Substituting (56)-(59) to (65), we obtain

$$\left\{\frac{u}{c_0} + \frac{c^*}{c_0}\left[\frac{1}{2} + \frac{a^2k^2g}{2(g+2s)}(1+a^2k^2)\right]\right\}\left\{\frac{c^*}{c_0}a^2k^2\frac{g}{2(g+2s)}(1+a^2k^2)\right\}\left(\frac{c^*}{c_0} + \frac{u}{c_0}\right)^2$$
$$= \left[\frac{1}{2} + \frac{a_0^2k_0^2g}{2(g+2s_0)}(1+a_0^2k_0^2)\right]a_0^2k_0^2\frac{g}{2(g+2s_0)}(1+a_0^2k_0^2), \tag{66}$$

where c^*/c_0 can be calculated by (63). Because s and $\hat{s}$ are functions of ak, equation (66) describes the relation between ak and u/c_0, and it is shown as Figure 3.

5 Waves on a beach

In this section, we consider the waves on a beach. We recall that the full modulation equations are

$$\frac{\partial A}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad A = \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial \omega}, \quad F = -\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial k}, \tag{67}$$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(HV) = 0, \quad V = U + \frac{kA}{H}, \tag{68}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial \overline{\zeta}}{\partial x} + \frac{1}{H}\frac{\partial S}{\partial x} = 0, \tag{69}$$

$$S = k(F - VA) + \overline{L}^{(\mathrm{w})} - H\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial H}, \tag{70}$$

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \tag{71}$$

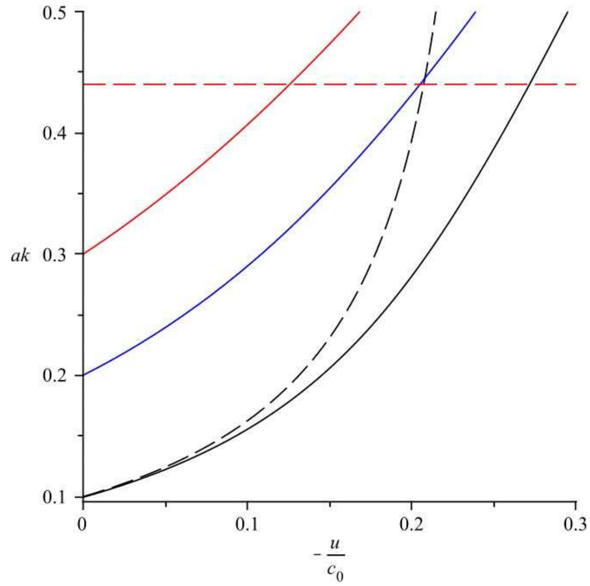

Figure 3: Wave steepness ak versus u/c_0; $a_0k_0 = 0.1, 0.2, 0.3$ (black, blue,red); wave breaking criterion $ak = 0.44$ (red dash) yields breaking for $|u|/c_0 > 0.27, 0.21, 0.13$. The dash line is the linear solution for $a_0k_0 = 0.1$.

where

$$\overline{L}^{(\mathrm{w})} = \frac{DE^*}{2} + \frac{D_2 k^2 E^{*2}}{2g} + O(E^{*3}), \tag{72}$$

and

$$D = \frac{\omega^{*2}}{gk\tanh(kH)} - 1, \quad \omega^* = \omega - Uk. \tag{73}$$

The mean momentum equation (69) has been rewritten.

Suppose that $h = h(x) \to 0$ as $x \to 0$, and that there is no background current. Then the steady solution ($\frac{\partial}{\partial t} = 0$) of these modulation equations yields the dispersion relation (71, 73), constant frequency $\omega = \omega_0$, and constant wave action flux and zero mass transport,

$$-\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial k} = F_0, \quad V = U + \frac{kA}{H} = 0, \quad \omega^* = \omega_0 - Uk. \tag{74}$$

Thus there is a mean Eulerian flow $U = -kA/H$, opposing the Stokes drift due to the waves. The mean momentum equation (69) then yields the wave set-up $\overline{\zeta}$,

$$g\frac{\partial \overline{\zeta}}{\partial x} + \frac{1}{H}\frac{\partial S}{\partial x} = 0, \quad S = kF_0 + \overline{L}^{(\mathrm{w})} - H\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial H}. \tag{75}$$

From (74), S as known in terms of H, and so

$$g\overline{\zeta} = -\int^x \frac{1}{H}\frac{\partial S}{\partial x}dx = -\int^H \frac{1}{H}\frac{\partial S}{\partial H}dH. \tag{76}$$

To illustrate, first make the small amplitude approximation. Then $\omega^* \approx \omega_0$, $H \approx h$, so that the dispersion relation becomes $\omega_0^2 = gk\tanh(kh)$ and yields $k = k(h)$, $D = 0$. The constant wave action flux condition reduces to

$$c_g a^2 = c_{g0} a_0^2, \tag{77}$$

where subscript "0" indicates the values at the depth $h = h_0$ offshore. The expression (76) can be written as follows. At first, we consider the total derivative of D. Now D can be considered as $D = D(k(h), h)$, so

$$\frac{dD}{dh} = \frac{\partial D}{\partial k}\frac{\partial k}{\partial h} + \frac{\partial D}{\partial h} = 0. \tag{78}$$

Because

$$\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial h} = \frac{E^*}{2}\frac{\partial D}{\partial h}, \quad \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial k} = \frac{E^*}{2}\frac{\partial D}{\partial k} = -F_0, \tag{79}$$

using (78) we obtain

$$\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial h} = F_0 \frac{\partial k}{\partial h}. \tag{80}$$

Using (80), $\frac{1}{h}\frac{\partial S}{\partial h}$ becomes

$$\begin{aligned}
\frac{1}{h}\frac{\partial S}{\partial h} &= \frac{1}{h}\left\{kF_0 - \frac{\partial}{\partial h}\left(h\frac{\partial \overline{L}^{(\mathrm{w})}}{\partial h}\right)\right\}, \\
&= \frac{1}{h}\left\{F_0\frac{\partial k}{\partial h} - \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial h} - h\frac{\partial^2 \overline{L}^{(\mathrm{w})}}{\partial h^2}\right\}, \\
&= -\frac{\partial^2 \overline{L}^{(\mathrm{w})}}{\partial h^2}.
\end{aligned}$$

Then, (76) becomes

$$\begin{aligned}
g\overline{\zeta} &= -\int^h \frac{1}{h}\frac{\partial S}{\partial h}dh, \\
&= \int^h \frac{\partial^2 \overline{L}^{(\mathrm{w})}}{\partial h^2}dh, \\
&= \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial h}, \\
&= -\frac{1}{2}\left(\frac{1-\tanh^2(kh)}{\tanh(kh)}\right)kE^*, \\
&= -\frac{1}{2}\left(\frac{-\sinh^2(kh)+\cosh^2(kh)}{\sinh(kh)\cosh(kh)}\right)kE^*, \\
&= -\frac{kE^*}{\sinh(2kh)},
\end{aligned} \tag{81}$$

so that,

$$\overline{\zeta} = -\frac{ka^2}{2\sinh(2kh)}, \tag{82}$$

where $\overline{\zeta}_0 = 0$. This is always negative, and so is a set-down. In shallow water $kh \to 0$, $c_g \approx (gh)^{1/2}$, and

$$\frac{k}{k_0} \approx \left(\frac{h_0}{h}\right)^{1/2}, \quad \frac{a}{a_0} \approx \left(\frac{h_0}{h}\right)^{1/4}, \tag{83}$$

so that, (82) can be approximated by

$$\overline{\zeta} \approx -\frac{a^2}{4h} = \frac{a_0^2 h_0^{1/2}}{4h^{3/2}}. \tag{84}$$

Since this small-amplitude theory predicts infinite amplitudes as $h \to 0$, we must consider nonlinear effects. One option is to impose an empirical wave-breaking condition $a/h = 0.44$ (Thornton and Guza, 1982, 1983), which defines the depth $h = h_b$, beyond which there is a surf zone. Here, we shall examine nonlinear effects in $h > h_b$ in the shallow water approximation $kH \to 0$. Then the Lagrangian (72) becomes

$$\overline{L}^{(\mathrm{w})} \approx \frac{DE^*}{2} - \frac{9E^{*2}}{16gk^2H^4}, \quad D \approx \frac{\omega^{*2}}{gHk^2}\left(1 + \frac{k^2H^2}{3}\right) - 1. \tag{85}$$

This Lagrangian is only valid when $ak \ll k^3H^3$, that is, for a very small Stokes number (Stokes, 1847). Using the linear shallow water expressions (83) we require that $S_0 = a_0/k_0^2h_0^3 \ll (h/h_0)^{9/4}$, which must fail as $h \to 0$. Hence, we infer that in shallow water we need to use a new theory, valid for Stokes number of order unity, so we consider the Korteweg-de Vries model next.

The Korteweg-de Vries (KdV) equation for weakly nonlinear long water waves, propagating on a constant undisturbed mean depth H, is given by (Mei, 1983, chapter 11.5.3)

$$\frac{\partial \zeta}{\partial t} + c_0\frac{\partial \zeta}{\partial x} + \frac{3c_0}{2H}\zeta\frac{\partial \zeta}{\partial x} + \frac{c_0H^2}{6}\frac{\partial^3 \zeta}{\partial x^3} = 0, \quad c_0 = (gH)^{1/2}. \tag{86}$$

The KdV balance has linear dispersion, represented by $H^3\frac{\partial^3\zeta}{\partial x^3}$, balanced by nonlinearity, represented by $\zeta\frac{\partial \zeta}{\partial x}$. To leading order, the waves propagate unchanged in form with the linear long wave speed $c_0 = (gH)^{1/2}$. Nonlinearity leads to wave steepening, opposed by wave dispersion, resulting in the KdV balance and the well-known solitary wave

$$\zeta = a_s \mathrm{sech}^2[\kappa(x - ct)], \quad \frac{c}{c_0} - 1 = \frac{a_s}{2H} = \frac{2\kappa^2H^2}{3}. \tag{87}$$

The periodic wave solution of KdV equation (86) is

$$\zeta = 2a\left[b(m) + cn^2(\gamma\theta; m)\right], \quad \omega = -\frac{\partial \theta}{\partial t}, \quad k = -\frac{\partial \theta}{\partial x}, \tag{88}$$

$$b = \frac{1-m}{m}\frac{E(m)}{mK(m)}, \quad \frac{a}{H} = \frac{2}{3}m\gamma^2(kH)^2, \quad \gamma = \frac{K(m)}{\pi}, \tag{89}$$

$$c = \frac{\omega}{k} = c_0 \left\{ 1 + \frac{a}{H} \left[\frac{2-m}{m} - \frac{3E(m)}{mK(m)} \right] \right\}, \tag{90}$$

Here $cn(x;m)$ is the elliptic function of modulus m where $0 < m < 1$, and $K(m), E(m)$ are elliptic integrals of the first and second kind. The amplitude is a and the mean value is 0. As $m \to 1$, this solution becomes a solitary wave, since then $b \to 0$ and $cn^2(x) \to \text{sech}^2(x)$. As $m \to 0$, $\gamma \to 1/2$, and it reduces to sinusoidal waves of small amplitude $a \sim m$. This cnoidal wave (88) contains two free parameters; we take these to be the amplitude a and the wavenumber k.

We now use the cnoidal wave expression (88) to evaluate the averaged Lagrangian (12), incorporating a mean current U,

$$\overline{L}^{(\mathrm{w})} = \left(\frac{c^{*2}}{gH} - 1 \right) G(m) \frac{E^*}{2} + O(E^{*2}), \quad E^* = \frac{ga^2}{2} \tag{91}$$

where

$$G(m) = 8(< cn^4(\gamma\theta; m) > -b^2), \tag{92}$$

or

$$G(m) = \frac{8[EK(4-2m) - 3E^2 - K^2(1-m)]}{3K^2m^2}. \tag{93}$$

To leading order the phase speed $c^* = W = (gH)^{1/2}$, while the wave action density, wave action flux and radiation stress now become, to leading order

$$A = \frac{\partial \overline{L}^{(\mathrm{w})}}{\partial \omega} = \frac{G(m)E^*}{\omega^*}, \quad F = -\frac{\partial \overline{\varpi}^{(\mathrm{w})}}{\partial k} = (U + c^*)A, \tag{94}$$

$$S = \frac{3\omega^* A}{2} = \frac{3G(m)E^*}{2}. \tag{95}$$

As before, we now seek the steady solutions ($\frac{\partial}{\partial t} = 0$), so again $\omega = \omega_0$ is the constant wave frequency, and to leading order $kh^{1/2} = k_0 h_0^{1/2}$ is constant. Next $F = F_0$ is the constant wave action flux, implying that, to leading order in wave amplitude,

$$h^{1/2} G(m) a^2 = \text{constant}. \tag{96}$$

Then using the expression (89) we find that $a \propto mK^2k^2h^3$ and so finally we get that

$$\tilde{G}(m) = K^4 m^2 G(m) = \text{constant} \cdot h^{-9/2}. \tag{97}$$

The wave amplitude determined from (96, 97) is shown in Figure 4. As $m \to 0$, $G \propto 1$, $\tilde{G} \propto m^2$, and so $m \propto h^{-9/4}$, $a \propto h^{-1/4}$ which is the linear Green's law (Green, 1837) result. But, as $m \to 1$, $G \propto K^{-1}$, $\tilde{G} \propto K^3$, $a \propto h^{-1}$.

Wave set-up is found from (69, 95) and is given by

$$g\overline{\zeta} = -\frac{1}{h}\frac{\partial S}{\partial x}, \quad S = \frac{3\omega A}{2} = \frac{3G(m)E^*}{2}. \tag{98}$$

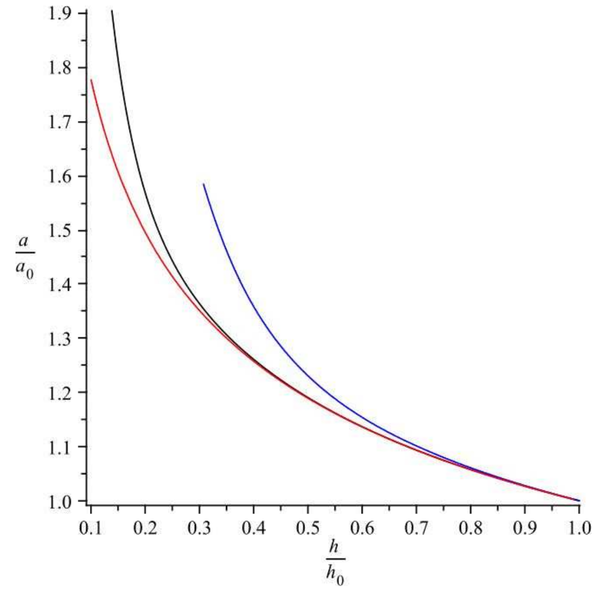

Figure 4: The wave amplitude is determined from (96, 97). The plots are for an initial modulus $m_0 = 0.1, 0.5$ (black, blue), while the linear solution $\zeta \propto h^{-1/4}$ is the red curve.

But since the wave frequency $\omega = kc_0$, $c_0 = (gh)^{1/2}$ and the wave action flux c_0A are conserved (see (96)), we readily find that

$$\overline{\zeta} = -\frac{a^2G(m)}{4h} = -\frac{a_0 2h_0^{1/2}G(m_0)}{4h^{3/2}}, \tag{99}$$

This is just the linear law again, and is independent of how the wave amplitude varies. Note that for $a_0/h_0 \ll 1$, $m_0 \approx 0$, $G(m_0) \approx 1$.

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