# GFD 2013 Lecture 1: Introduction to Gravity Currents

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## 1 Introduction

Gravity currents can be found in various contexts, including weather and climate (e.g. sea breezes, haboobs), pyroclastic flows and spills (e.g. Liquefied Natural Gas), and are perhaps most obvious upon opening the front door on a cold day (Fig. 1). This also explains why

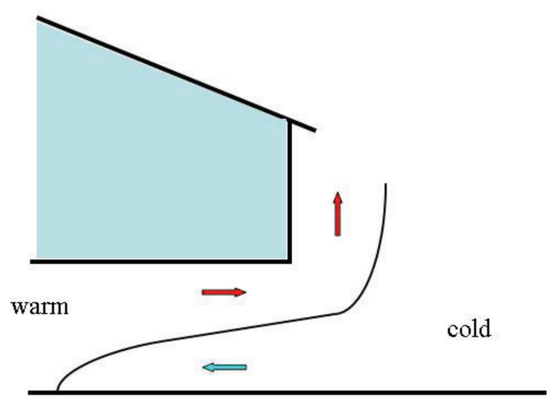


Figure 1: A sketch of a house after the door had been opened. The warm air escapes from the house through a gravity current along the ceiling, while cold air from the outside enters near the floor.

opening the door after taking a hot shower in order to clear the mirror is a futile attempt. The warm and moist air escapes near to the ceiling, so that the bottom half of the bathroom is clear, leaving the top part of the mirror unusable for shaving. A good understanding of gravity currents allows us to predict the speed of lava flows and haboobs, the strength of sea breezes, and many other useful properties of these phenomena. In addition, gravity currents have some important technical applications.

### 1.1 Lock exchange

Lock exchange is the classic experiment for studying gravity currents. One begins with a tank that is filled with two fluids of different densities that are separated by a barrier (see

Fig. 2). Upon removal of the barrier, the denser fluid flows along the bottom boundary into the lighter fluid, and there is a return flow in the lighter fluid. This process is driven by

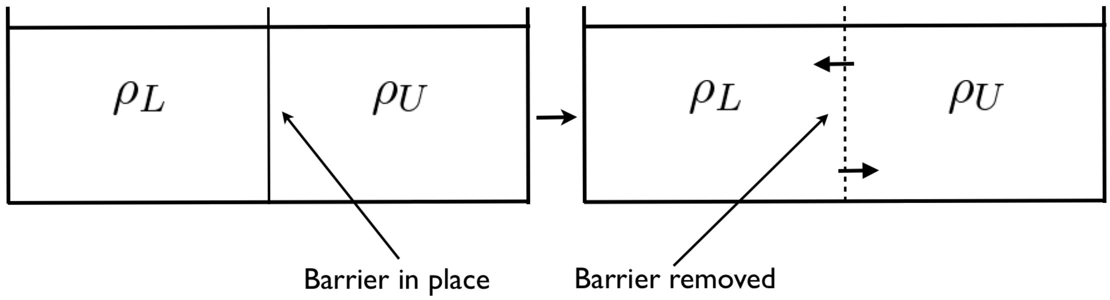


Figure 2: The two fluids are initially separated by a barrier. Once the barrier is removed, two gravity currents develop (arrows). In this setup, $\rho_U < \rho_L$.

a horizontal pressure gradient that arises from the density differences. Depending on the viscosities of the two fluids, the gravity current is subject to entrainment.

### 1.2 Reduced gravity

The buoyancy force acting on a body of density $\rho_b$ that is immersed in a surrounding fluid of density $\rho_f$ can be derived from Archimedes principle. This principle tells us that the force, $\mathcal{F}$, acting on the body is given by the gravity force acting on the body minus the gravitational force arising from the displacement of the surrounding fluid.

$$\mathcal{F} = -g(\rho_b - \rho_f) \tag{1}$$

If the density of the body is greater, the body will sink, and otherwise it will move upward. We can then define the effective gravitational constant, known as 'reduced gravity' as

$$g' = \frac{g\,(\rho_b - \rho_f)}{\rho_b}. \tag{2}$$

Switching to the variables used in the lock exchange experiment described earlier, we have

$$g' = \frac{g\,(\rho_L - \rho_U)}{\rho_L}. \tag{3}$$

This expression will be useful when we analyze the time evolution of gravity currents, because it is the gravitational acceleration used in defining the velocity scale.

### 1.3 Equations of motion

Below, we introduce the governing equations for most of the examples in these notes.

### 1.3.1 Navier-Stokes Equations

The governing equations for fluid in a box are found by enforcing mass and momentum conservation. They read

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\, \rho \mathbf{u} \,) = 0 \tag{4a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}. \tag{4b}$$

Here, $\rho$ is the density field, $\mathbf{u}$ is the velocity field, $p$ is the pressure field, $\mathbf{g}$ is the gravitational acceleration, and $\nu$ is the kinematic viscosity of the fluid ($\nu \approx 10^{-6}$ m$^2$/s for water at 20 °C).

For most cases, we are going to assume incompressibility of the fluid under consideration, i.e.,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \Leftrightarrow \nabla \cdot \mathbf{u} = 0. \tag{5}$$

Relation (5) together with Eq. (4b) are known as the incompressible Navier-Stokes equations.

### 1.3.2 Vorticity Equation

By taking the curl of Eq. (4b), we can recast the Navier-Stokes equations in vorticity form, in which $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Applying the identity $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega}$ to this equation gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \underbrace{\frac{1}{\rho^2} (\, \nabla \rho \times \nabla p \,)}_{\text{baroclinic term}} + \nu \nabla^2 \boldsymbol{\omega}. \tag{6a}$$

The fluid is *barotropic* if the density is a function of pressure only, so that isocontours of pressure and density are aligned and the baroclinic term in Eq. (6a) vanishes. If pressure and density isocontours are not aligned, the baroclinic term is nonzero, the flow is *baroclinic*.

### 1.3.3 Boussinesq Approximation

The Boussinesq approximation neglects density differences in the momentum equations except where they multiply the gravitational acceleration. Mathematically speaking, this approximation is the first order correction in an expansion in the density difference between a background profile and the actual density profile. We decompose the density field into a constant density, $\rho_0$, and small spatio-temporal variations $\rho^*$ so that

$$\rho(\mathbf{x}, t) = \rho_0 + \rho^*(\mathbf{x}, t) \tag{7a}$$

$$\rho^*(\mathbf{x}, t) = \overline{\rho}(z) + \rho'(\mathbf{x}, t). \tag{7b}$$

Here, we assumed that $\rho_0 =$ const. and $|\rho^*| \ll \rho_0$. Eq. (7b) defines a further decomposition of the small density variations $\rho^*$ into a field $\overline{\rho}$ that varies only in the vertical and a field $\rho'$, which is a function of both space and time. The pressure field can be decomposed in a similar way, giving

$$p(\mathbf{x}, t) = -\rho_0 g z + \overline{p}(z) + p'(\mathbf{x}, t), \tag{8}$$

where $\overline{p}(z)$ is defined via the hydrostatic relation

$$\frac{d\overline{p}}{dz} = -\overline{\rho} g \tag{9}$$

Combining Eq. (7b), (8), the gravity and pressure terms from Eq. (4b) can be expanded in $\rho'$ and $p'$ yielding

$$-\frac{1}{\rho}\nabla p + \mathbf{g} \approx -\frac{1}{\rho_0}\nabla p' - \mathbf{g}\frac{\rho'}{\rho_0}. \tag{10}$$

## 1.4 Frontogenesis

First, consider an inviscid fluid with a constant horizontal density gradient, $\rho_x = \rho_{x0}$ (shown in Fig. 3a). From Eq. (5), we find

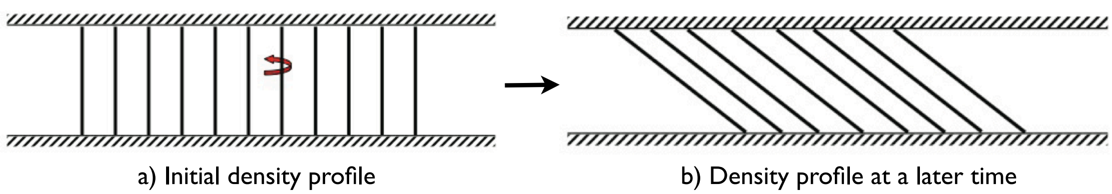


Figure 3: Lines of constant density for a) the initial state and b) at some later time. In both diagrams, the density is higher on the left than it is on the right.

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0 \qquad t = 0. \tag{11}$$

Differentiating with respect to $x$ gives

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{\partial \rho}{\partial x} = -\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x} = \frac{\partial w}{\partial z}\frac{\partial \rho}{\partial x}, \tag{12}$$

where the continuity condition, Eq. (5), was used to derive Eq. (12). Assuming a solution exists in which $w = 0$ and neglecting diffusivity, Eq. (4b) reduces to

$$u_t - \frac{1}{\rho_0}p_x = 0. \tag{13}$$

Using the hydrostatic condition, $p_z = -g\rho$, and integrating with respect to $u$ and $z$ whilst assuming $\rho_x$ remains constant gives

$$u = \frac{g}{\rho_0}\rho_{x0} zt. \tag{14}$$

This is equivalent to a linear shear in the flow that grows linearly with time. The final state of the fluid is shown in Fig. 3b. Now consider a case in which there is a region of small density gradient and a region of large density gradient, as in Fig. 4a. $u$ grows faster in the region where $|\rho_x|$ is large than in the region where $|\rho_x|$ is small, so some fluid is

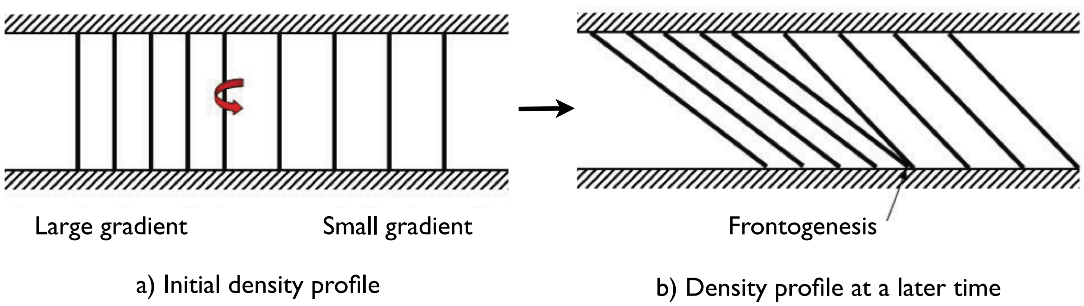


Figure 4: Lines of constant density. The horizontal density gradient on the left of each figure is larger than the density gradient on the right of each figure.

forced upwards between these two regions. $w$ must be nonzero because $u_x$ is nonzero. The Richardson number, Ri, where

$$\mathrm{Ri} = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}\left(\frac{\partial u}{\partial z}\right)^{-2}, \tag{15}$$

can be easily derived from Eq. (12) and Eq. (14), giving Ri= $\frac{1}{2}$ for all time.

### 1.4.1 Rotating the tank

The results of this analysis were tested in an experiment by Simpson and Linden [1]. Setting up a horizontally stratified fluid like the one in Fig. 3a is difficult, and so they set up a fluid in which the isopycnals were already tilting, as in Fig. 3b. This was achieved by stratifying a long thin tank which was then rotated by 90°. The angle of the isopycnals that are produced in this process can be estimated by modeling the box as a long and thin ellipse and then solving Poisson's equation in a rotating reference frame. We start by moving from the laboratory frame to the rotating frame of the tank, giving an initial vorticity of $-2\Omega$, where $\Omega$ is the rotation rate of the tank. Conserving vorticity, we can write Possion's equation for the stream function $\psi$,

$$\nabla^2\psi = -2\Omega. \tag{16}$$

Modeling the tank as an ellipse with vanishing flow on the solid boundaries suggests a trial solution of

$$\psi = c\left(1 - \left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2\right), \tag{17}$$

where $x$ and $y$ are in the rotating reference frame and $a$ and $b$ are the axes of the ellipse. Eq. (17) can now be substituted into Eq. (16) to determine the constant value of $c$. Taking the limit $b \ll a$ (i.e. the limit of a long thin tank), gives

$$\psi \approx \Omega b^2\left(1 - \left(\frac{y}{b}\right)^2\right). \tag{18}$$

From Eq. (18), we calculate the velocity profile, and hence find the distance $x$ a particle would travel in the flow field during the time $\Delta t$ is

$$x = u\Delta t = 2\Omega y \Delta t. \tag{19}$$

Rotating the tank from a vertical initial position to the horizontal requires $\Omega \Delta t = \frac{\pi}{2}$, so that the angle between the isopycnals and the vertical is given by

$$\phi = \arctan\left(\frac{x}{y}\right) = \arctan(\pi) \approx 72^\circ. \tag{20}$$

### 1.5 Yih's Analysis (1947)

Yih [3] derived an expression for the velocity of gravity currents for the simple case of two fluids in a finite box separated by a barrier. Removing the perfect barrier instantaneously

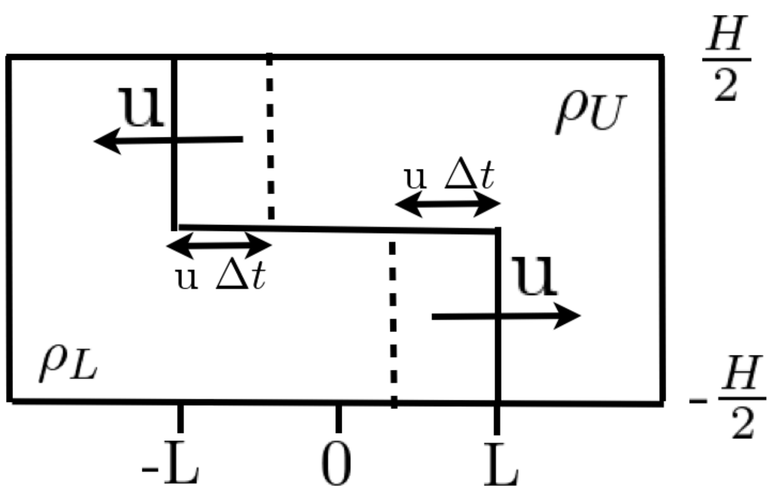


Figure 5: Schematic of Yih's model.

and neglecting frictional effects, the two gravity currents develop symmetrically, as in Fig. 5. Assuming that they travel at a velocity $U$ and that they occupy half of the box height each, we can derive a simple expression for the change in potential energy over the infinitesimal time $\Delta t$, giving

$$\Delta \mathrm{PE} = \frac{1}{4} g' \rho_L H^2 U \Delta t. \tag{21}$$

At the same time, the kinetic energy of the system changes due to changes in both the velocity of the currents and the amount of mass in them

$$\Delta \mathrm{KE} = \frac{1}{2} (\rho_L + \rho_U) H U^3 \Delta t. \tag{22}$$

In the absence of energy loss, the change in potential energy must balance the change in kinetic energy, $\Delta\mathrm{PE} = \Delta\mathrm{KE}$. This allows us to derive an expression for $U$ in terms of densities $\rho_U$ and $\rho_L$

$$U^2 = \frac{1}{4} g' H, \; g' = \frac{g(\rho_L - \rho_U)}{(\rho_L + \rho_U)}. \tag{23}$$

The dimensionless velocity or Froude number for this flow is

$$F_h = \frac{U}{\sqrt{g' H}} = \frac{1}{2}. \tag{24}$$

It is found to be about 0.46–0.47 in experiments with Yih's setup.

### 1.6 von Kármán's analysis (1940)

Von Kármán's [2] analysis differs from Yih's analysis in that the gravity current occurs in an infinite fluid. Assuming irrotational, inviscid fluid and a frame of reference in which the front is stationary, we get the setup shown in Fig. 6. Because we assume that the fluid is

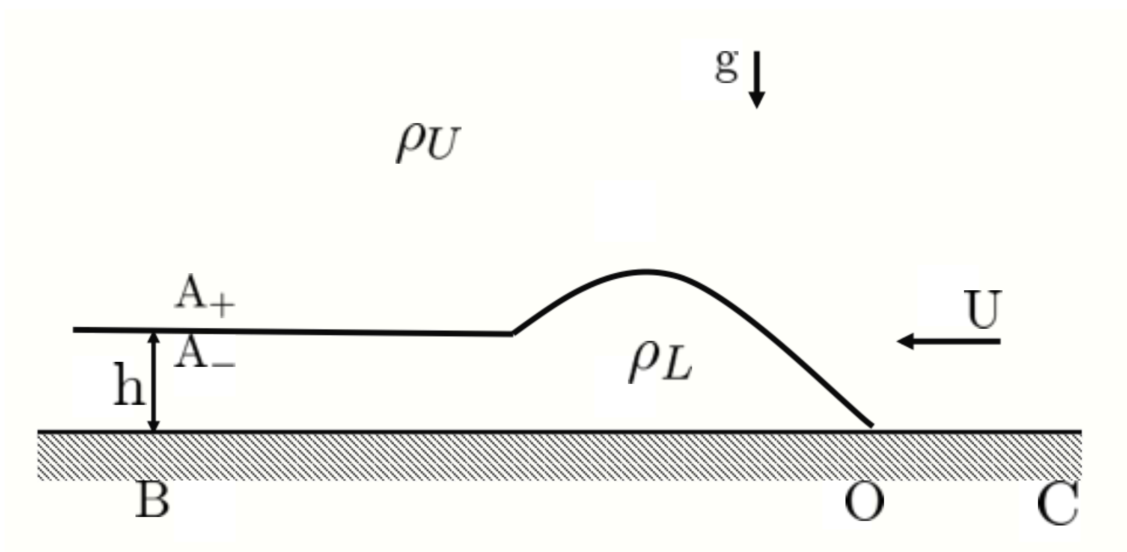


Figure 6: Sketch of the setup for von Kármán's analysis, in which the frame of reference is such that the front is stationary. $U$ is the speed of the fluid. $A$ is the position at the density interface where $\frac{\partial \rho}{\partial x} = 0$. $B$ is below it $A$ on the bottom boundary. $O$ is the position at which the density interface intersects the boundary, and $C$ is some distance away, again on the boundary.

irrotational, we can apply Bernoulli's equation along streamlines. Taking the streamline between $O$ and $A_+$ gives

$$p_0 = p_A + g\rho_U h + \frac{1}{2}\rho_U U^2, \tag{25}$$

and taking the streamline between $O$ and $A_-$ gives

$$p_o = p_A + g\rho L h. \tag{26}$$

Subtracting Eq. (25) from Eq. (26), we can solve for $U^2$, giving

$$U^2 = \frac{2g\left(\rho_L - \rho_u\right)h}{\rho_U}. \tag{27}$$

Now we can redefine $g'$ as

$$g' = \frac{g\left(\rho_L - \rho_U\right)}{\rho_L}, \tag{28}$$

giving a Froude number of

$$F_h = \frac{U}{\sqrt{g'h}} \tag{29a}$$

$$= \sqrt{\frac{2}{\gamma}}, \tag{29b}$$

where $\gamma = \frac{\rho_U}{\rho_L}$. As $\gamma \to 0$, $F_h$ becomes infinite, because this equation only applies in the Boussinesq limit. If we take this limit (i.e. $\gamma = 1$), $F_h = \sqrt{2}$.

## References


[1] J. E. Simpson and P. F. Linden, *Frontogenesis in a fluid with horizontal density gradients*, J. Fluid Mech., 202 (1989), pp. 1–16.

[2] von Kármán, *The engineer grapples with nonlinear problems*, Bull. Am. Math. Soc., 46 (1940), pp. 615 – 683.

[3] C. S. Yih, *A study of the characteristics of gravity waves at a liquid interface*, master's thesis, State University of Iowa, 1947.