Lecture 2: Concepts from Linear Theory

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1 Introduction

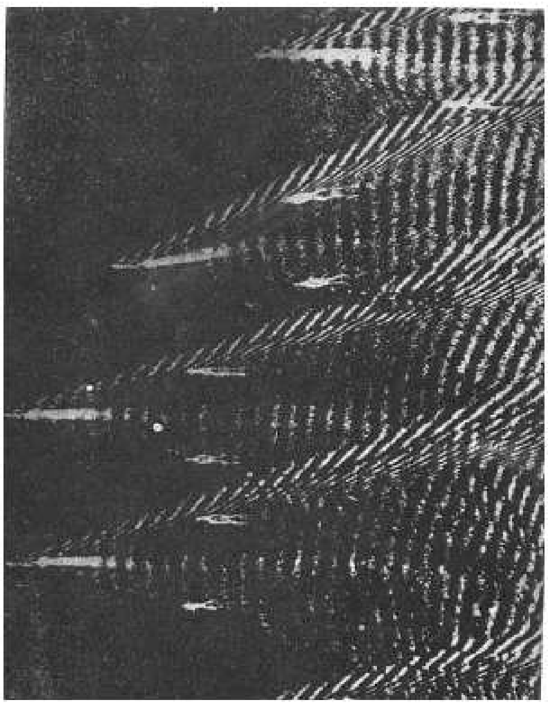

Fig. 5.9 – Ship waves on deep water. Photographing a model gives a more dramatic pattern than a full size ship. If the boat is not obviously a model the capillary waves in front show the scale clearly.

Figure 1: Ship waves from WW II battleships shown on the left (from *Water Waves* by Stoker [3]) and a toy boat on the right from (*Introduction to Water Waves* by Crapper [1]). Kelvin's (1887) method [2] of stationary phase predicts both.

Figure 1 shows the wave patterns observed in the wake of ships and boats, which are among the few kinds of waves that are well-explained by linear theory. In this lecture, we study linear theory by starting from the nonlinear equations of water waves derived in Lecture 1, linearizing them, and exploring some fundamental concepts which emerge from the linearized equations such as the phase and group velocities and particle orbits. The predictions from linear theory also help distinguish between cases where surface tension effects are negligible (gravity waves) and where they dominate (capillary waves). Different characteristics of shallow and deep water waves (such as particle paths in each case) are also pointed out.

2 Nonlinear equations of motion

Starting from nonlinear equations of motion for an irrotational flow with no wind forcing derived in Lecture 1, we know that

$$\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi, \tag{1}$$

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{\sigma}{\rho}\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} \right), \tag{2}$$

have to be satisfied at the surface, defined as $z = \eta(x, y, t)$. The first relation is the kinematic boundary condition while the second one is the Bernoulli's law expressing the dynamic boundary condition at the surface. For the interior flow, the Laplace equation

$$\nabla^2\phi = 0 \qquad\qquad -h(x, y) < z < \eta(x, y, t), \tag{3}$$

needs to be satisfied and the bottom boundary condition is

$$\partial_z \phi + \nabla\phi \cdot \nabla h = 0, \qquad \text{on} \quad z = -h(x, y). \tag{4}$$

For small amplitude waves, the nonlinear terms in (1) and (2) can be neglected to linearize the problem around the state of rest. Further simplifications can be achieved by arguing that to first order the boundary conditions can be evaluated at $z = 0$ rather than $z = \eta$. This can be justified by expanding (1) and (2) in a Taylor series around $z = 0$ and keeping linear terms only. For example the $\partial_z\phi$ in (1) yields

$$\frac{\partial\phi}{\partial z}\,|_{z=\eta} = \frac{\partial\phi}{\partial z}\,|_{z=0} + \eta\frac{\partial^2\phi}{\partial z^2}\,|_{z=0} + \ldots \simeq \frac{\partial\phi}{\partial z}\,|_{z=0}\,. \tag{5}$$

Applying these simplifications and assuming a flat bottom (h=const) leads to the following set of linearized equations:

$$\partial_t \eta = \partial_z \phi, \qquad \text{on} \quad z = 0, \tag{6}$$

$$\partial_t \phi + g\eta = \frac{\sigma}{\rho}\nabla^2\eta \qquad \text{on} \quad z = 0, \tag{7}$$

$$\nabla^2\phi = 0 \qquad -h < z < 0, \tag{8}$$

and

$$\partial_z \phi = 0, \qquad \text{on} \qquad z = -h. \tag{9}$$

The goal is to solve (8) for the interior flow subject to the conditions (6), (7), and (9). As a warm-up problem, let us first find solutions to (8) assuming that ϕ is known on the upper boundary. For example, if

$$\phi(x, y, 0, t) = a \sin(kx), \tag{10}$$

at the top boundary and (9) at the bottom boundary, the solution is

$$\phi(x, y, z, t) = a \frac{\cosh(k(z + h))}{\cosh(kh)} \sin(kx). \tag{11}$$

If the top boundary condition is changed to

$$\phi(x, y, 0, t) = a \sin(kx) \cos(ly), \tag{12}$$

then

$$\phi(x, y, z, t) = a \frac{\cosh(\kappa(z + h))}{\cosh(\kappa h)} \sin(kx) \cos(ly), \tag{13}$$

where $\kappa^2 = k^2 + l^2$.

For simplicity, the problem can be rewritten in complex variables. Hence if the function ϕ takes the form

$$\phi(x, y, 0, t) = \Re\{a\ e^{ikx+ily}\} \qquad \text{on} \qquad z = 0. \tag{14}$$

then

$$\phi(x, y, z, t) = \Re\left\{a \frac{\cosh(\kappa(z + h))}{\cosh(\kappa h)}\ e^{ikx+iky}\right\}. \tag{15}$$

in the whole domain. Finally, note that if ϕ is time-dependent on the boundary, it can be expressed as a linear combination of components in the form of (14), so that the solution will be the equivalent linear combination of the respective solutions (15):

$$\phi(x, y, z, t) = \Re\left\{\iiint \Phi(k, l, \omega) \frac{\cosh(\kappa(z + h))}{\cosh(\kappa h)}\ e^{ikx+ily-i\omega t} \mathrm{d}k\, \mathrm{d}l\, \mathrm{d}\omega\right\}, \tag{16}$$

leading to an equivalent expression for η:

$$\eta(x, y, t) = \Re\left\{\iiint H(k, l, \omega)\ e^{ikx+ily-i\omega t} \mathrm{d}k\, \mathrm{d}l\, \mathrm{d}\omega\right\}. \tag{17}$$

Next, (16) and (17) can be used to satisfy the boundary conditions (6) and (7) at the surface to find the linearized dispersion relation for the frequency ω:

$$\omega^2 = \left(g + \frac{\sigma}{\rho}\kappa^2\right) [\kappa \cdot \tanh(\kappa h)], \qquad \kappa^2 = k^2 + l^2. \tag{18}$$

Since the expression obtained for ω depends on the wave number, it is called a “dispersion” relation. If a system of linear evolution equations has a dispersion relation, then that relation encodes all of the information about wave propagation from the partial differential equations (but not from their boundary or initial conditions).

From this point on, we restrict our attention to 2-D motions. For a linearized 2-D case ($\partial_y \equiv 0$), the dispersion relation (18) reduces to

$$\omega^2 = \left(gh + \frac{\sigma}{\rho}k^2 h\right)\left[\frac{\tanh(kh)}{kh}\right] \cdot k^2, \tag{19}$$

and we may choose,

$$\omega(k) = \sqrt{gh + \frac{\sigma}{\rho}k^2 h} \cdot \sqrt{\frac{\tanh(kh)}{kh}} \cdot k. \tag{20}$$

The general solution of the 2-D linear problem can then be written in the form of a sum over all components like those in (16) and (17):

$$\begin{aligned}\phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_-(k)\frac{\cosh(k(z+h))}{\cosh(kh)}e^{i(kx-\omega(k)t)}\mathrm{d}k \,+ \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_+(k)\frac{\cosh(k(z+h))}{\cosh(kh)}e^{i(kx+\omega(k)t)}\mathrm{d}k,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\omega(k)}{g+\frac{\sigma}{\rho}k^2}\Phi_-(k)e^{i(kx-\omega(k)t)}\mathrm{d}k \,- \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\omega(k)}{g+\frac{\sigma}{\rho}k^2}\Phi_+(k)e^{i(kx+\omega(k)t)}\mathrm{d}k.\end{aligned} \tag{22}$$

The first integrals in (21) and (22) correspond to waves traveling to the right and the second integrals correspond to leftward-travelling waves.

3 Phase velocity and group velocity

The "phase velocity" for each wavenumber k in (22) is defined by

$$c_p(k) = \frac{w(k)}{k} \geq 0, \tag{23}$$

where $\omega(k)$ was defined by the dispersion relation (20). The phase speed is the velocity with which the crest of each wave travels. If the phase speed varies with k, each wave travels with its own velocity and they progressively disperse. In contrast if all waves travel with the same velocity, then the system is called "non-dispersive". Sound waves are examples of non-dispersive waves.

Another velocity can be defined which is called the "group velocity":

$$c_g = \frac{\mathrm{d}w}{\mathrm{d}k}. \tag{24}$$

This is the velocity with which the energy of the waves propagates and is not necessarily equal to the wave's phase velocity. To see this, consider adding two waves with slightly different wavenumbers $k, k + \delta k$ $(\delta k \ll k)$ and frequencies $\omega(k), \omega(k + \delta k) \approx \omega(k) + \frac{d\omega}{dk} \cdot \delta k$. The combination of the two waves has the form

$$\eta(x,t) = \sin\{kx - \omega t\} + \sin\{(k + \delta k)x - (\omega + c_g \delta k)t\}$$

$$= 2\sin\{(k + \frac{\delta k}{2})x - (\omega + \frac{c_g \delta k}{2}t\} \cdot \cos\left\{\frac{\delta k}{2}x - \frac{c_g \delta k}{2}t\right\} \tag{25}$$

$$\Rightarrow \quad \eta(x,t) \approx 2\sin\{k(x - c_p t)\} \cdot \cos\left\{\frac{\delta k}{2}(x - c_g t)\right\}. \tag{26}$$

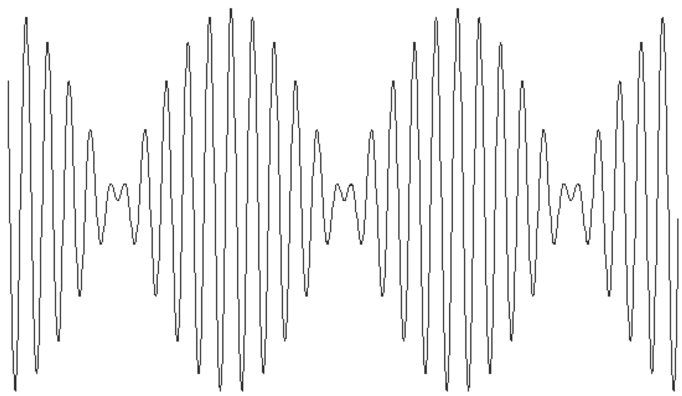

Figure 2: Formation of a wave group by linear combination of two waves

The $\sin(k(x - c_p t))$ term corresponds to the fast oscillations (shown in Figure 2) moving with the phase speed $c_p = \omega/k$. The amplitude of this wave is modulated by a slowly varying function $\cos\{\frac{\delta k}{2}(x - c_g t)\}$ which has a longer wavelength $4\pi/\delta k$ and propagates at a speed $c_g = d\omega/dk$. Multiplication of a rapidly varying and a slowly varying wave in (25) leads to periodic wave groups (Figure 2). The individual wave components travel with the phase speed c_p while the envelope of the wave group travels with the group speed c_g. For a wave with a phase speed larger than the group speed, the wave crests appear from behind the wave envelope at the nodal point, progress forward, and disappear at the front nodal point. For $c_p < c_g$ the reverse is true. Thus, following the crest of the largest wave in a wave envelope shows that it only dominates the group for a limited time due to the slow modulation. For this reason, a surfer would not be able to enjoy riding a large wave for a long time unless the phase and group velocities were almost the same. In this scenario, the large wave almost holds its shape as it travels with the group. Since no energy can pass through the nodal points, the wave energy can only travel with the group velocity and not the phase velocity.

4 Gravity waves

In most oceanographic applications, surface tension is negligible. In this section, we therefore restrict our attention to the case where the surface tension does not play an important role, so that equation (20) reduces to

$$\omega(k) = \sqrt{gh} \cdot \sqrt{\frac{\tanh(kh)}{kh}} \cdot k, \tag{27}$$

which is the dispersion relation for "gravity waves". Figure 3 shows the plot of the frequency ω versus k. The slope of the straight line connecting a point on the curve and the origin is the phase velocity ω/k while the slope of the tangent to the curve at that point is the group velocity $\mathrm{d}\omega/\mathrm{d}k$. The bottom panel shows the phase velocity and the group velocity curves. The peak of the phase speed curve corresponds to the fastest moving wave. For gravity waves, the longest waves travel fastest and with the maximum speed of $\sqrt{gh}$. This can be obtained from (27) by letting $kh \to 0$ or by calculating the slope of the dispersion curve in the $kh \to 0$ limit. It is important to note that the maxima of the phase and group velocity curves do not necessarily coincide.

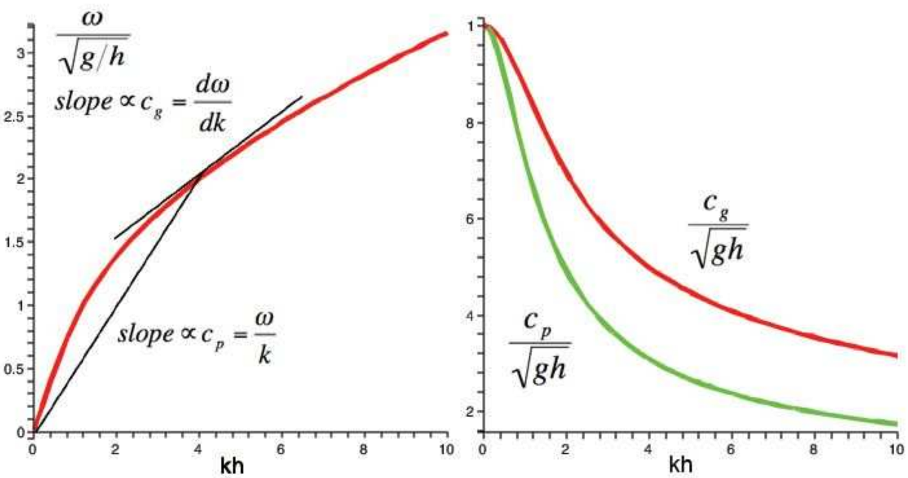

Figure 3: (left) Determining the group and phase velocities from the dispersion curve. (right) Phase and group velocity.

To have an idea of the maximum phase speed of gravity waves in nature, the maximum wave speed in the bay of Bengal (a potential tsunami region with a depth of 3500 m) is 185 m/s (ie. 670 km/hr=415 mi/hr). The deepest point in the ocean (near Guam) is around 11,000m deep which is deeper than the height of Mt. Everest (8848 m). The fastest gravity wave in the ocean can have a velocity of 328 m/s (1182 km/hr = 734 mi/hr). Although this is very close to speed of sound in the air, it is considerably slower than the speed of sound in water which is 1450 m/s (at $10^o C$).

Finally, note that two limiting cases can be considered to simplify the dispersion relation (27). For $h/\lambda \gg 1$ (or $hk \gg 1$), the dispersion relation and its corresponding phase velocity reduce to

$$\omega = \sqrt{gk}, \tag{28}$$

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \tag{29}$$

since $\tanh(x) \to 1$ for $x \to \infty$. This is called the "Deep Water" approximation and implies that deep-water waves are dispersive. It should be noted that kh needs not be very large for this approximation to hold as $\tanh(kh) = 0.94138$ for only $kh = 1.75$ and $\tanh(kh) = 0.999993$ for $kh = 2\pi$. In the "Shallow Water" limit on the other hand, we have $kh \ll 1$. Since $\tanh(x) \to x$ as $x \to 0$, the dispersion relation and phase velocity relations for gravity waves take the form of

$$\omega = \sqrt{gh} \cdot k \tag{30}$$

$$c_p = c_g = \sqrt{gh}, \tag{31}$$

which implies that shallow water waves are approximately non-dispersive and all travel with the same speed at a similar depth. This makes waves in shallow enough waters good candidates for surfing, a fact well-known by surfers.

5 Waves with surface tension

The dispersion relation (20) includes the surface tension and can be used to find the general expression for the phase speed:

$$c_p = \sqrt{\left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)\tanh(kh)} = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}\right)\tanh(\frac{2\pi h}{\lambda})}, \tag{32}$$

where $\lambda = 2\pi/k$ is the wavelength. Figure 4 shows the plot of the phase speed versus the wavelength to demonstrate the effect of surface tension. As seen in equation (32), surface tension increases the phase speed for all wavenumbers, a fact which can easily be interpreted on physical grounds since tension increases the restoring force on the free surface. However, Figure 4 shows that this effect is only noticeable for short waves ($\lambda < \lambda_{\min}$). For deep water waves($\tanh(kh) \simeq 1$), the phase velocity takes the form

$$c_p = \frac{\omega}{k} = \frac{\sqrt{g + \frac{\sigma}{\rho}k^2}}{\sqrt{|k|}} \tag{33}$$

and the wavelength of the slowest wave ($c_{\min}$) can be obtained by setting $\mathrm{d}c_p/\mathrm{d}\lambda = 0$ yielding

$$c_{\min} = \left[\frac{4g\sigma}{\rho}\right]^{1/4} \quad \text{at} \quad \lambda_{\min} = 2\pi\sqrt{\frac{\sigma}{\rho g}}. \tag{34}$$

For air-water interface at 20^{o}C, $c_{\min}$ and $\lambda_{\min}$ take the values of 23.2 cm/s and 1.73 cm respectively. As the figure shows, this implies that waves with wavelengths of a few centimeters or less are affected by surface tension. These waves are called "capillary" waves and are often referred to as "ripples". For wavelengths of few millimeters surface tension is the dominating factor with the effect of gravity being negligible. For waves longer than a few centimeters gravity plays the dominant role. As mentioned earlier and also shown on the figure, gravity waves have a maximum speed of $\sqrt{gh}$. As the figure shows, for every

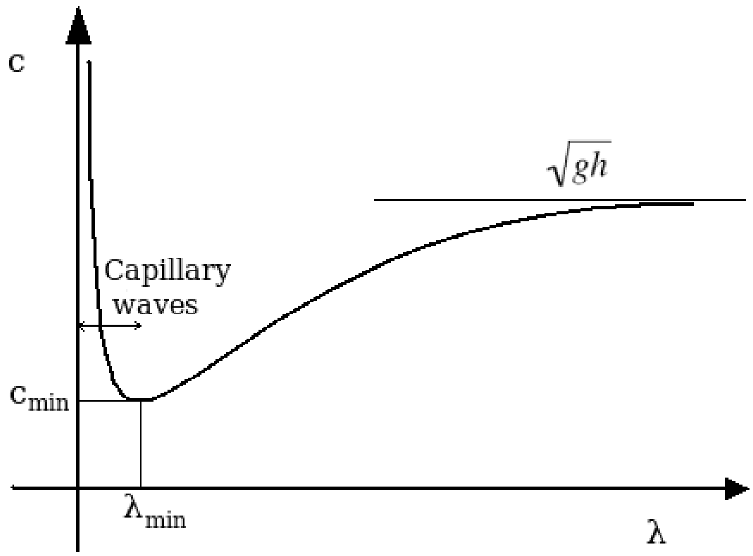

Figure 4: Phase velocity vs wavelength for surface gravity waves.

(long) gravity wave, there is a (short) capillary wave with the same phase speed. Remote sensing of ocean waves depends on this fact. One can also plot the graph of the group velocity versus the wavelength and observe that the group velocity also has a minimum, at a wavelength slightly different from that of the phase velocity.

It should also be noted that there are capillary waves which can travel faster than the fastest gravity waves although they are very small and rapidly dissipate in the presence of viscosity. Another point shown on the figure is that for gravity waves, long waves travel faster than short waves while the opposite is true for capillary waves. To show this, Figure 5 demonstrates the propagation of gravity waves (left) and capillary waves (right) in deep water. Clearly, long gravity waves travel faster at the outer front of the wave trains in the left panel while the shorter capillary waves travel faster and form the front in the right panel. Since the effect of surface tension on capillary waves guarantees a minimum group speed, the innermost waves seen in the right panel of Figure 5 (darkest ring) are the ones which travel with the minimum possible group velocity and form the inner boundary of the wave pattern. This leaves a flat interior region.

6 Particle paths

In this section, we investigate fluid particle paths for a wave traveling to the right with a constant phase speed $c_p(k)$. Assuming the rest position of the particle to be (x_0, z_0), and its instantaneous position to be $(x = x_0 + x_1, z = z_0 + z_1)$, the equations of motion for the

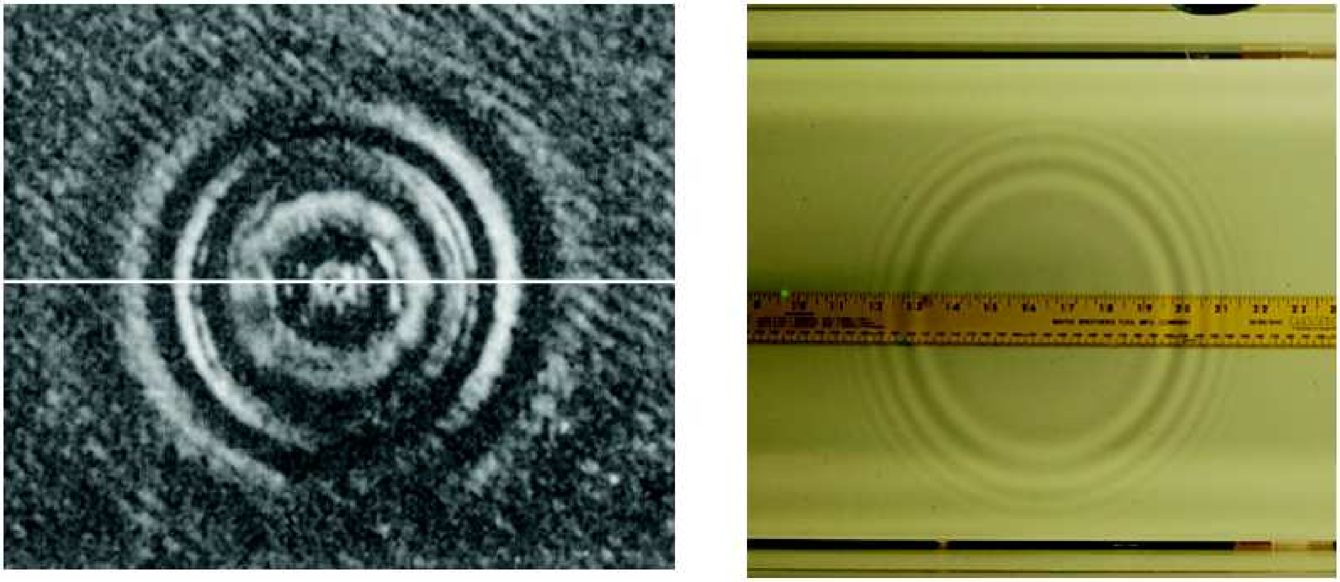

Figure 5: Propagation of surface gravity waves on the left (from *Water Waves* by Stoker [3]) and capillary waves on the right (Courtesy of Diane Henderson).

particle are:

$$\frac{\mathrm{D}x}{\mathrm{D}t} = \frac{\mathrm{D}x_1}{\mathrm{D}t} = u(x,z,t) = a\cos(kx-\omega(k)t)\frac{\cosh(k(z+h))}{\cosh(kh)}, \tag{35}$$

$$\frac{\mathrm{D}z}{\mathrm{D}t} = \frac{\mathrm{D}z_1}{\mathrm{D}t} = w(x,z,t) = a\sin(kx-\omega(k)t)\frac{\sinh(k(z+h))}{\cosh(kh)}. \tag{36}$$

Note that these two equations describe the motion of the fluid particle based on a velocity field obtained from linear theory. The approximate particle paths can be obtained using these ODEs by making some additional simplification. For small-amplitude waves, one can assume the particle displacement (x_1, z_1) away from its neutral position (x_0, z_0) to be small. The particle velocity is then the same as that of the neutral position to linear order so that

$$\frac{\mathrm{D}x_1}{\mathrm{D}t} = u(x,z,t) = a\cos(kx_0-\omega(k)t)\frac{\cosh(k(z_0+h))}{\cosh(kh)}, \tag{37}$$

$$\frac{\mathrm{D}z_1}{\mathrm{D}t} = w(x,z,t) = a\sin(kx_0-\omega(k)t)\frac{\sinh(k(z_0+h))}{\cosh(kh)}. \tag{38}$$

Integrating in time we get

$$x_1 = x(t) - x_0 \approx -\frac{a}{\omega}\sin(kx_0-\omega(k)t)\frac{\cosh(k(z_0+h))}{\cosh(kh)}, \tag{39}$$

$$z_1 = z(t) - z_0 \approx \frac{a}{\omega}\cos(kx_0-\omega(k)t)\frac{\sinh(k(z_0+h))}{\cosh(kh)}. \tag{40}$$

The above two relations can be combined to give

$$\frac{(x(t)-x_0)^2}{\cosh^2(k(z_0+h))}+\frac{(z(t)-z_0)^2}{\sinh^2(k(z_0+h))}=\frac{a^2}{\omega^2} \tag{41}$$

which indicates that the particle orbits are ellipses to the first order. It should be noted that for deep water waves the orbits are nearly circular as $\cosh(k(z_0+h)) \approx \sinh(k(z_0+h))$ for $kh \gg 1$. Meanwhile, for shallow water waves the orbits become thin ellipses with their major axes almost independent of the depth. Figure 6 shows the motion of marked fluid particles over one wave period. Orbits are nearly circular near the top of this flow, progressively becoming thin ellipses towards the bottom boundary. Stokes [4] showed that at second order, there is a slow drift near the free surface (Stokes drift). The drift is visible in Figure 6 by inspecting some of the larger orbits near the top.

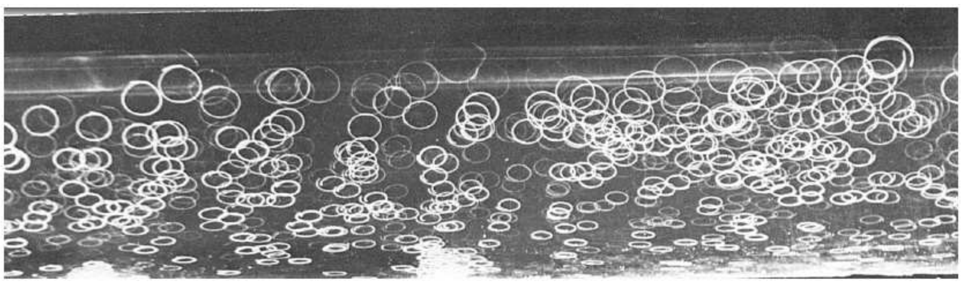

Figure 6: Motion of marked fluid particles over one wave period From [5].

References

[1] G. Crapper, *Introduction to water waves*, Ellis Horwood Ltd., (1984).

[2] L. W. T. Kelvin, Phil Mag., 4 (1887), pp. 252–255.

[3] J. Stoker, *Water waves: The mathematical theory with applications*, John Wiley & Sons, Inc., (1992).

[4] G. G. Stokes, *On the theory of oscillatory waves*, Transactions of the Cambridge Philosophical Society, 8 (1847), pp. 441–473.

[5] F. Wallet, A & Ruellan, Houille Blanche, 5 (1950), pp. 483–489.