Convection, Stability and Turbulence

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1 Introduction

The three basic forms of heat transport are:

- Radiation
- Conduction
- Convection

In many physical systems, all of these processes may play a role (see Figures 1–2 below). Convection is buoyancy-driven flow due to density variations in a gravitational field. It is particularly important as it enhances the vertical heat transport. For example, without convection, it would take many hours to heat a saucepan of water to boiling point, simply because heat diffusion in water is so weak ($\kappa \approx 1.5 \times 10^{-3}\text{cm}^2\,\text{s}^{-1}$).

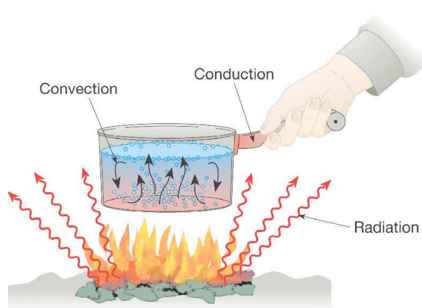

Figure 1: Heat transfer in a saucepan.

2 Rayleigh-Bénard convection

In 1900, Henri Bénard [2] performed an experiment concerning convection cells in a thin liquid layer. He observed spontaneous pattern formation - the convection was seen to organise itself into hexagonal cells throughout the entire domain, as shown in Figure 3.

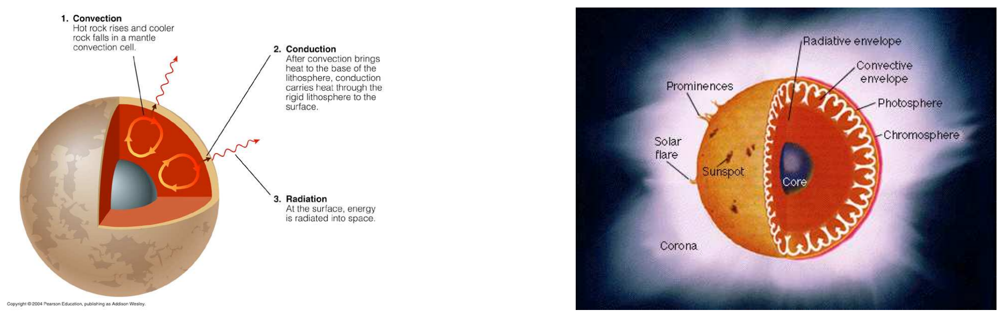

Figure 2: Heat transfer in the Earth and Sun.

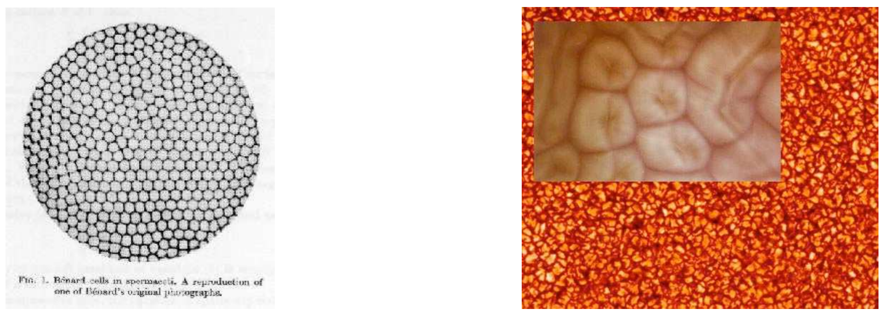

Figure 3: Convection cells in Bénard's experiment and on the surface of the Sun.

Rayleigh [28] was the first to undertake a mathematical theory of fluid convection. His analysis described the formation of convection rolls for fluid confined between parallel plates held at different temperatures. The width of the rolls at convective onset is proportional to the depth of the fluid layer, but also depends on details of the boundary conditions.

However, these convection rolls are a different physical phenomenon to the convection cells observed by Bénard. Such convection cells are now known to be associated with Marangoni flows, where variations of surface tension with temperature play a crucial role. In fact, there are several regimes of convection. As the temperature difference is increased, spiral defect chaos may be observed, before finally giving way to fully turbulent convection, with the associated breakdown of spacial–temporal correlations (see Figure 4).

2.1 A mathematical theory of convection

The theory first developed by Rayleigh is now known as Rayleigh–Bénard convection. Provided that the temperature variations within the fluid layer remain small, we may adopt the Boussinesq approximation, which neglects compressibility except for the presence of a buoyancy term in the momentum equation. Furthermore, it may be shown that viscous heating is often negligible compared to thermal driving from the boundary conditions. The

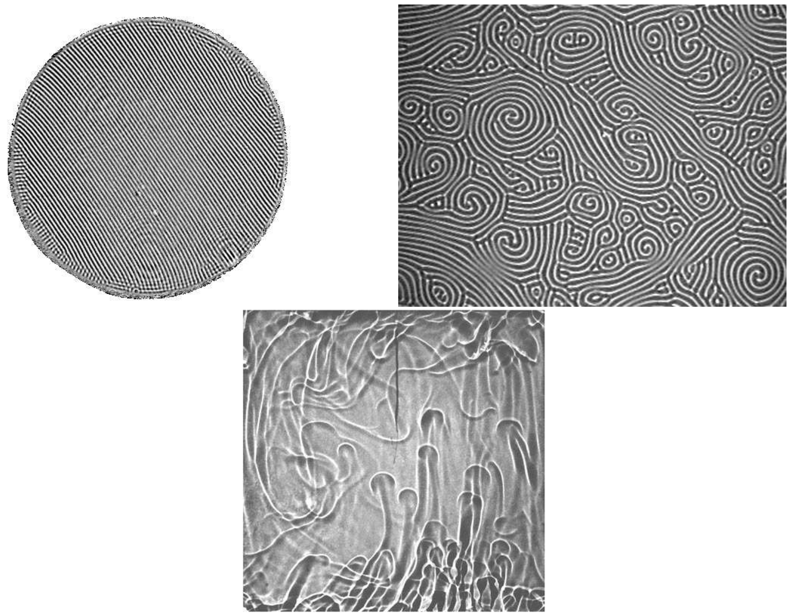

Figure 4: Convection rolls give way to complex spatio–temporal dynamics, and eventually to fully-developed turbulent convection. (The last figure is a side view.)

equations of mass, momentum and heat conservation are

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{u}, & (1) \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho_0} \nabla (p - p_0) &= \nu \nabla^2 \mathbf{u} + g\alpha \hat{\mathbf{k}} (T - T_0), & (2) \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \kappa \nabla^2 T. & (3) \end{aligned}$$

The last equation arises from the energy equation

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{4}$$

where $q = cT$ is the thermal energy density and $\mathbf{J} = c(\mathbf{u}T - \kappa \nabla T)$ is the heat flux. In equation (2) ν is kinematic viscosity, g is acceleration due to gravity, c is specific heat capacity and α is the thermal expansion coefficient, which also appears in the equation of state

$$\frac{\rho - \rho_0}{\rho_0} = \alpha (T - T_0). \tag{5}$$

Rayleigh considered a domain confined between two impermeable flat horizontal plates, each held at a different temperature. The distance between the plates is h, and the problem is assumed to be periodic in the two horizontal directions, x and y (see Figure 5). We are primarily concerned with the mean vertical heat flux, namely

$$\begin{aligned} \langle J_z \rangle &= \left\langle wcT - c\kappa \frac{\partial T}{\partial z} \right\rangle & (6) \\ &= \frac{c\kappa}{h} (T_{\text{hot}} - T_{\text{cold}}) + c \langle wT \rangle & (7) \end{aligned}$$

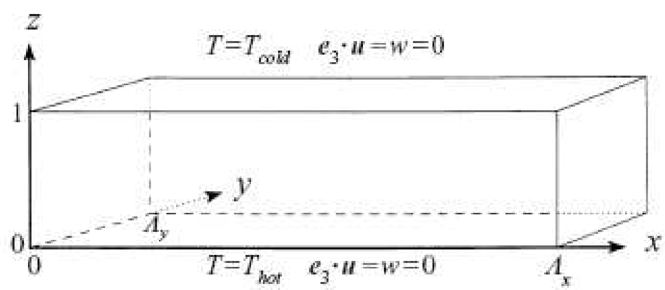

Figure 5: The mathematical domain.

where $\langle\cdot\rangle$ denotes a space-time average.

From this point onwards, we shall work with non-dimensionalised equations. The characteristic length scale of the system is the height h, and the characteristic time scale is the thermal diffusion time over this distance, h^2/κ. We therefore formally define the following dimensionless variables:

$$\tilde{\mathbf{x}} = \mathbf{x}/h \tag{8}$$

$$\tilde{t} = \kappa t/h^2 \tag{9}$$

$$\tilde{\mathbf{u}} = \mathbf{u}h/\kappa \tag{10}$$

$$\tilde{T} = \frac{T - T_0}{T_{\mathrm{hot}} - T_{\mathrm{cold}}} \tag{11}$$

$$\tilde{p} = \frac{p - p_0}{\rho_0}\frac{h^2}{\kappa^2} \tag{12}$$

If we further define the dimensionless parameters

- Rayleigh number, $\mathrm{Ra} = \dfrac{g\alpha(T_{\mathrm{hot}} - T_{\mathrm{cold}})h^3}{\nu\kappa}$
- Prandtl number, $\mathrm{Pr} = \nu/\kappa$

then the governing equations become (after dropping the '$\sim$'s)

$$0 = \nabla \cdot \mathbf{u}, \tag{13}$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) + \nabla p = \nabla^2 \mathbf{u} + \mathrm{Ra}\, \hat{\mathbf{k}} T, \tag{14}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T \tag{15}$$

The boundary conditions on T are now simply $T = 1$ on $z = 0$ and $T = 0$ on $z = 1$. Finally, we define the Nusselt number, Nu, which measures the enhancement of heat transport due to convection, relative to conductive heat transfer:

$$\mathrm{Nu} = \langle J_z \rangle / J_{\mathrm{cond}} \tag{16}$$

We wish to determine how Nu varies with the parameters Ra and Pr[1]. From the equations of motion, it can be shown (see Appendix A) that

$$\begin{aligned} \mathrm{Nu} &= 1 + \langle wT \rangle & (17)\\ &= \left\langle |\nabla T|^2 \right\rangle & (18)\\ &= 1 + \frac{1}{\mathrm{Ra}} \left\langle |\nabla \mathbf{u}|^2 \right\rangle \;\geq\; 1, & (19) \end{aligned}$$

the last expression showing that convection always *increases* the rate of heat transfer. If we chose to represent the convective heat flux in terms of an effective "eddy diffusion" κ_{eddy}, so that $\langle J_z \rangle = c\kappa_{\mathrm{eddy}}(T_{\mathrm{hot}} - T_{\mathrm{cold}})/h$, we would then find that $\kappa_{\mathrm{eddy}} = \mathrm{Nu}\ \kappa$. In the case of no convection we have the "conduction solution", $\mathbf{u} = 0$, $T = \tau(z) = 1 - z$, with $\mathrm{Nu} = 1$.

2.2 Linear instability

As we increase the temperature difference between the two plates (i.e. increase the Rayleigh number) we expect the conduction solution to become unstable. We therefore analyse the linear stability of the system by defining a perturbation θ such that $T = \tau(z) + \theta(\mathbf{x}, t)$. We now linearise our equations in order to obtain a single linear equation for θ:

$$(-\tfrac{1}{\mathrm{Pr}}\partial_t + \nabla^2)(-\partial_t + \nabla^2)\nabla^2\theta - \mathrm{Ra}(\nabla^2 - \partial_z^2)\theta = 0. \tag{20}$$

We look for solutions of the form $\theta \propto \mathrm{e}^{-\lambda t + \mathrm{i}k_x x + \mathrm{i}k_y y}$, which yields the 1D eigenfunction problem

$$(\tfrac{1}{\mathrm{Pr}}\lambda + \partial_z^2 - k^2)(\lambda + \partial_z^2 - k^2)(\partial_z^2 - k^2)\theta + \mathrm{Ra}\ k^2\theta = 0, \tag{21}$$

where $k^2 = k_x^2 + k_y^2$. On both plates we have $\theta = 0$ and $\partial_z^2\theta = 0$ so that $w = 0$. If we further assume, for simplicity, that we have no-stress boundary conditions on the two plates, then the solutions of (21) are $\theta \propto \sin(n\pi z)$ for $n \in \mathbb{Z}$. We find that $\lambda \in \mathbb{R}\ \forall k, n$, so the condition for marginal stability is $\lambda = 0$. The marginally stable modes have

$$\mathrm{Ra} = \frac{(k^2 + n^2\pi^2)^3}{k^2}, \tag{22}$$

and we find the critical Rayleigh number Ra_c (above which the system is unstable to disturbances of some wavelength) by minimising this expression over k and n. Therefore, regardless of the choice of Pr, we find $\mathrm{Ra}_c = \frac{27}{4}\pi^4 \approx 657$, at which point convective rolls appear with width $\pi/k_c = \sqrt{2}h$. If we instead choose no-slip boundary conditions on the two plates, then the eigenfunctions become more complicated, and the critical Rayleigh number increases to $\mathrm{Ra}_c \approx 1708$.

2.3 Nonlinear stability

Linear stability does not rule out the possibility that the system might become nonlinearly unstable at Rayleigh numbers smaller than Ra_c. In order to determine sufficient conditions for *stability*, we can perform an "energy analysis", retaining all terms in the equations. We

[1] In a domain of finite horizontal extent, the cross-sectional area would also enter as a parameter.

again suppose that $T = \tau(z) + \theta(\mathbf{x}, t)$, where $\tau(z) = 1 - z$, and consider perturbations that are periodic in x and y. By taking the scalar product of $\mathbf{u}$ with (14) and applying (13) we find

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2\,\mathrm{Pr}}\right) + \nabla\cdot\left(\frac{u^2}{2\,\mathrm{Pr}}\mathbf{u} + p\mathbf{u}\right) = \nabla^2(\tfrac{1}{2}u^2) - |\nabla\mathbf{u}|^2 + \mathrm{Ra}\;Tw. \tag{23}$$

Integrating over the domain and applying the boundary conditions, we find

$$\frac{1}{\mathrm{Pr}\,\mathrm{Ra}}\frac{d}{dt}\int_V \tfrac{1}{2}u^2\,dV = -\frac{1}{\mathrm{Ra}}\int_V |\nabla\mathbf{u}|^2\,dV + \int_V \theta w\,dV. \tag{24}$$

Similarly, we can multiply (15) by $\theta = T - \tau(z)$ to obtain

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\theta^2) + \nabla\cdot(\tfrac{1}{2}\theta^2\mathbf{u}) = -(\tau' w - \tau'')\theta + \nabla^2(\tfrac{1}{2}\theta^2) - |\nabla\theta|^2, \tag{25}$$

which becomes, after integration,

$$\frac{d}{dt}\int_V \tfrac{1}{2}\theta^2\,dV = -\int_V |\nabla\theta|^2\,dV - \int_V (\tau' w - \tau'')\theta\,dV. \tag{26}$$

Since $\tau(z) = 1 - z$, the sum of equations (24) and (26) is

$$\frac{dE}{dt} = -Q \tag{27}$$

$$= -(Q/E)E \;\le\; -\left(\min_{\theta,\mathbf{u}}[Q/E]\right)E \tag{28}$$

where

$$E\{\theta,\mathbf{u}\} = \int_V \frac{1}{2}\left[\frac{u^2}{\mathrm{Pr}\,\mathrm{Ra}} + \theta^2\right]\,dV \tag{29}$$

and

$$Q\{\theta,\mathbf{u}\} = \int_V \left[\frac{|\nabla\mathbf{u}|^2}{\mathrm{Ra}} + |\nabla\theta|^2 - 2\theta w\right]\,dV. \tag{30}$$

If we can show that $\min_{\theta,\mathbf{u}}[Q/E]$ is strictly greater than zero (for given Ra), then equation (28) implies that E tends exponentially to zero, so the system is *absolutely stable*. Since both E and Q are quadratic forms, it suffices to minimise over θ and $\mathbf{u}$ such that $E\{\theta,\mathbf{u}\} = 1$. We therefore seek to extremise the functional

$$\mathcal{F}\{\theta,\mathbf{u}\} = Q\{\theta,\mathbf{u}\} - 2\lambda E\{\theta,\mathbf{u}\} - \int_V \frac{2}{\mathrm{Ra}}p(\mathbf{x})\nabla\cdot\mathbf{u}\,dV \tag{31}$$

where λ and $p(\mathbf{x})$ are, respectively, global and pointwise Lagrange multipliers, designed to ensure that $E = 1$ and $\nabla\cdot\mathbf{u} = 0$ throughout the domain. The Euler–Lagrange equations for this system are

$$-\frac{\lambda}{\mathrm{Pr}}\mathbf{u} + \nabla p = \nabla^2\mathbf{u} + \mathrm{Ra}\;\theta\hat{\mathbf{k}} \tag{32}$$

$$-\lambda\theta - w = \nabla^2\theta \tag{33}$$

From these, it can be shown that $Q\{\theta, \mathbf{u}\} = 2\lambda$, so it just remains to show that λ is positive. In fact, we can combine (32–33) into the single equation

$$(\tfrac{1}{\mathrm{Pr}}\lambda + \nabla^2)(\lambda + \nabla^2)\nabla^2\theta - \mathrm{Ra}(\nabla^2 - \partial_z^2)\theta = 0. \tag{34}$$

Comparing this to equation (20), we see that the nonlinear stability criterion is *identical* to the linear stability criterion.

In principle, we could repeat this analysis for a different functional form of $\tau(z)$ in equation (26). We shall see in §4.1 that this approach allows us to derive upper bounds for the Nusselt number Nu when $\mathrm{Ra} > \mathrm{Ra}_c$.

3 Heat transport at high Ra

In the decades following Rayleigh's discovery, most studies of convection focused on the weakly nonlinear regime at Rayleigh numbers just above critical. The first author to study fully turbulent convection was Malkus [18]. Later, Kraichnan [16] applied mixing length theory to predict the behaviour of the Nusselt number in the limit of large Rayleigh number.

A more extensive theory was proposed by Grossmann & Lohse [12], which led to a complicated picture with eight separate parameter regimes, depending on the values of Ra and Pr (see Figure 6). The "ultimate" scaling obtained in the limit of large Rayleigh number

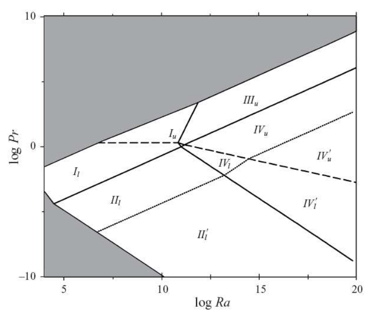

Figure 6: The various parameter regimes predicted by [12].

was predicted by both Kraichnan and Grossmann & Lohse to be $\mathrm{Nu} \sim (\mathrm{Pr}\,\mathrm{Ra})^{1/2}$.

3.1 The "ultimate" scaling: $\mathrm{Nu} \propto (\mathrm{Pr}\,\mathrm{Ra})^{1/2}$

In fact, the "ultimate" scaling law can be derived from the so-called "free-fall" argument of Spiegel [31]. If the diffusivities are sufficiently small, then the rate of heat flux may be determined by the bulk dynamics. The vertical velocity scale is therefore set by balancing inertia and buoyancy acceleration:

$$w^2/h \sim g\alpha\delta T. \tag{35}$$

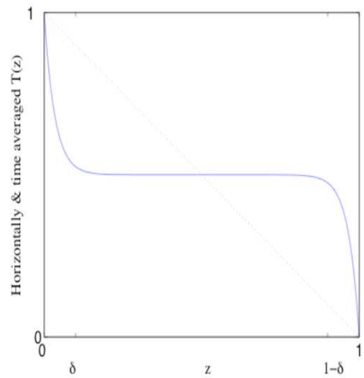

Figure 7: A sketch of the mean temperature in a turbulently-convecting Rayleigh-Bénard cell.

Since the thermal energy density is $c\delta T$, the effective vertical heat flux is then $J_{conv} \sim c\delta T w$, which is independent of ν and κ. Thus

$$\begin{aligned} \mathrm{Nu} &\sim \frac{J_{conv}}{J_{cond}} && (36)\\ &\sim \frac{c\Delta T[g\alpha h\Delta T]^{1/2}}{c\kappa\Delta T/h} && (37)\\ &\sim (\mathrm{Pr}\,\mathrm{Ra})^{1/2}. && (38) \end{aligned}$$

3.2 The "classical" scaling: $\mathrm{Nu} \propto \mathrm{Ra}^{1/3}$

An argument in favour of the scaling $\mathrm{Nu} \propto \mathrm{Ra}^{1/3}$ was originally proposed by Malkus [19], and quantitatively articulated by Howard [13]. Howard's argument, which appeals to a plume-like, time-dependent solution of the heat equation, is omitted here in favour of a simpler, though less precise, approach.

In a statistically steady state, the mean temperature in a turbulently-convecting cell should be uniform in the well-mixed interior, with most of the variation occuring in thin boundary layers of thickness δ at the top and bottom of the cell, as illustrated in Figure 7. In dimensionless variables, the heat flux (hence the Nusselt number Nu) away from the lower boundary will be proportional to the gradient of T there, namely δ^{-1}. In order to determine the boundary layer thickness, we assume *marginal stability.* That is, we assume that the Rayleigh number based on conditions in the boundary layer is (almost) exactly critical. Thus

$$\mathrm{Ra}_c \approx \mathrm{Ra}_\delta = \frac{\alpha g(\delta h)^3\Delta T}{2\nu\kappa} = \frac{1}{2}\delta^3\mathrm{Ra} \qquad (39)$$

$$\Rightarrow \delta \approx (2\mathrm{Ra}_c)^{1/3}\mathrm{Ra}^{-1/3} \qquad \Rightarrow \mathrm{Nu} \propto \mathrm{Ra}^{1/3}. \qquad (40)$$

An interesting aside is that the prediction made here for the prefactor is fairly accurate, lending further weight to this simple argument.

3.3 Experimental and numerical validation of the scaling laws

Recent years have seen several experimental works attempt to verify the various propsed scalings for heat transport at high Rayleigh number. In experiments with gaseous helium, Chavanne *et al.*[4] claimed to have observed a distinct high-Ra (in their case, $\mathrm{Ra} > 10^{11}$) scaling for Nu, perhaps matching the predictions of Kraichnan. Their results were later contradicted by Glazier *et al.* [11], who observed no deviation from another scaling, $\mathrm{Ra}^{2/7}$, in their experiments with liquid mercury, though the experimental Prandtl number was much smaller than in [4]. Experimental evidence for a $\mathrm{Ra}^{0.301} \approx \mathrm{Ra}^{3/10}$ scaling (or perhaps a $(\mathrm{Ra} \log \mathrm{Ra})^{3/10}$ scaling) was obtained by Niemela *et al.* [21]. High-precision experiments carried out by Nikolaenko *et al.*[22] suggest that the $\mathrm{Ra}^{1/3}$ scaling can be observed for convecting cells with large aspect ratio. While the debate over the correct scaling is by no means settled, the three-dimensional simulations of Amati *et al.* [1], reproduce the classical $\mathrm{Ra}^{1/3}$ scaling.

The somewhat ephemeral nature of the high-Ra scaling may perhaps be indicative of the fact that boundaries play a non-trivial role in determining turbulent heat transport, accentuating differences in experimental apparatus. A sufficiently rough boundary may disrupt any boundary layers, changing the behaviour of the system. Experiments in a rough-walled container by Roche *et al.* [29] produced evidence for the $\mathrm{Ra}^{1/2}$ scaling, a result that was reproduced by the numerical simulations of Stringano *et al* [32]. Notably, their simulation yielded *different* results when their jagged bottom boundary was replaced by a smooth, flat boundary.

4 Analytical upper bounds on Nu

With so many experiments giving different results, one would like to investigate the possibility of calculating mathematical upper bounds for the Nusselt number Nu. This can be done by using the integral identities for Nu (17-19) and applying standard inequalities from calculus in order to obtain bounds. One method of carrying out such a calculation is the so-called *background method*, which is outlined below.

4.1 The background method

We begin by allowing the temperature field T to take the form

$$T(\mathbf{x}, t) = \tau(z) + \theta(\mathbf{x}, t) \tag{41}$$

We are entirely free to choose $\tau(z)$, as long as it satisfies the inhomogeneous boundary conditions. Unlike in the energy analysis of stability described earlier in §2.3, it need not satisfy the steady-state conduction equation (with $\mathbf{u} = 0$). Unlike a linear stability analysis, we do not intend to linearise about $\tau(z)$, so a difference of any size between this and the 'true' steady background state may be included in the perturbation θ.

Using the above form of the temperature field, the equations of heat and momentum conservation become

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta + \tau'' - w\tau, \tag{42}$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathrm{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \nabla^2\mathbf{u} + \mathrm{Ra}(\tau+\theta)\hat{\mathbf{z}}, \tag{43}$$

with boundary conditions $\theta(0) = \theta(1) = 0$. Using the identity (18), and denoting by angled brackets a combined spatial/temporal average, we find that

$$\mathrm{Nu} = \left\langle |\nabla T|^2 \right\rangle = \int_0^1 [\tau'(z)]^2 \mathrm{d}z + 2\left\langle \tau'\frac{\partial\theta}{\partial z}\right\rangle + \left\langle |\nabla\theta|^2\right\rangle. \tag{44}$$

We can eliminate the term involving $\tau'\mathrm{d}\theta/\mathrm{d}z$ in (44) by multiplying the heat equation (42) by 2θ taking the time/space average and integrating by parts. The resulting equation is

$$\mathrm{Nu} = \int_0^1 [\tau'(z)]^2 \mathrm{d}z - \left\langle |\nabla\theta|^2 + 2\tau' w\theta\right\rangle \tag{45}$$

Finally, for convenience later, we add the time/space-average of $2\mathbf{u}\cdot(43)$ to (45) to find that

$$\mathrm{Nu} = \int_0^1 [\tau'(z)]^2 \mathrm{d}z - Q\{\theta, \mathbf{u}\}, \tag{46}$$

where

$$Q\{\theta, \mathbf{u}\} = \left\langle |\nabla\theta|^2 + \frac{2}{\mathrm{Ra}}|\nabla\mathbf{u}|^2 + 2(\tau'-1)w\theta\right\rangle. \tag{47}$$

This general result forms the backbone of the background method. If one can find a particular $\tau(z)$ such that the quadratic form Q is non-negative definite, then the first term on the right-hand side of (46) provides an upper bound on the Nusselt number Nu. An illustration of how one could perform such an analysis is given in detail in the Appendix.

One should not expect that there is a unique $\tau(z)$ that will give a positive definite quadratic form Q. Indeed, one may formulate a variational calculus problem based upon choosing $\tau(z)$ in order to minimise the upper bound subject to the temperature boundary conditions. This very problem has been solved, numerically, by Plasting & Kerswell [27].

It is interesting to note that the background method may be thought of a rigorous implementation of the marginal stability argument proposed in §3.2. For a hypothesised temperature profile (for example, the well-mixed interior with thin boundary layers in Figure 7), we aim to estimate the heat flux obtained as if this profile were an equilibrium solution.

4.2 A brief history of bounds on Nu

The background method is not the only analytical technique to have been applied to the problem of high-Ra convection. The original analysis by Howard [13] used a power balance combined with some statistical hypotheses about the nature of turbulent convection to arrive at the bound $\mathrm{Nu} < C\mathrm{Ra}^{1/2}$ uniformly in Pr. No author since has been able to improve the exponent of Ra for arbitrary (finite) Pr, lending support to the 'ultimate' scaling $\mathrm{Nu} \propto \mathrm{Ra}^{1/2}$. The prefactor was improved, however, through the asymptotics of

Busse [3]. This bound was originally unchallenged by the application of the background method Constantin & Doering [5], until Nicodemus *et al.* [20] introduced a so-called 'balance parameter' to the background method, allowing the use of largely-isothermal background temperature profiles. It was shown by Kerswell [15] that the best possible bounds obtainable by the background method and by Howard [13] should be one and the same. The most strict currently-known bound was determined by Plasting & Kerswell [27], who obtained the optimal profile for use in the background method, and arrived at the bound

$$\mathrm{Nu} < 0.0264\,\mathrm{Ra}^{1/2}. \tag{48}$$

4.3 Convection and heat transport at $\mathrm{Pr} = \infty$

Several authors have extended the the search for a high-Ra scaling to fluids with infinite Prandtl number - that is, fluids with very small thermal diffusivity, relative to their (kinematic) viscosity. A very important example of such a fluid is the Earth's mantle. In the infinite-Pr regime the equations of heat, momentum and mass conservation are

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T, \tag{49}$$

$$\nabla p = \nabla^2 \mathbf{u} + \mathrm{Ra}T\hat{\mathbf{z}}, \tag{50}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{51}$$

Note that the velocity and pressure fields are slaved to the temperature field T in this case. One can perform both a linear stability analysis and an energy analysis in exactly the same manner as for the finite Pr fluid case illustrated in §§2.2-2.3, arriving at precisely the same result. The flow is linearly unstable for $\mathrm{Ra} > \mathrm{Ra}_c$ and nonlinearly stable for $\mathrm{Ra} < \mathrm{Ra}_c$, with $\mathrm{Ra}_c = 1708$ for a problem with no-slip boundaries at the top and bottom of the cell, $\mathrm{Ra}_c = 27\pi^4/4$ with stress-free boundaries and so on. Note that, although the marginal stability conditions remain the same, behaviour above Ra_c *does* depend on Pr.

If one were to investigate the behaviour of the Nusselt number Nu at high Ra, however, one finds notable differences between this and the finite Pr case. It is still possible to find analytical upper bounds for Nu using the background method described above; the Nusselt number still takes the form

$$\mathrm{Nu} = \int_0^1 [\tau(z)]^2 \mathrm{d}z - Q\{\theta\} \tag{52}$$

but the quadratic form Q becomes

$$Q\{\theta\} = \left\langle |\nabla\theta|^2 + 2\tau' w\theta \right\rangle, \quad \text{where} \quad \nabla^2 w = -\mathrm{Ra}\nabla_H^2\theta. \tag{53}$$

Analyses have yielded a number of bounds on Nu, each specifying different power laws. The first application of the background method to this problem Doering & Constantin [9] yielded a bound of the form $\mathrm{Nu} \le C\mathrm{Ra}^{2/5}$. It was later proved, both by Otero [23] and by Plasting [26], that $\mathrm{Ra}^{2/5}$ is the best possible scaling that may be obtained with a monontone background $\tau(z)$, using the background method alone. Yan [33] improved the bound to $\mathrm{Ra}^{4/11}$ by instead using a maximum principle applied to a monotone profile. The addition of a singular integral analysis to this maximum principle can yield a scaling of the form

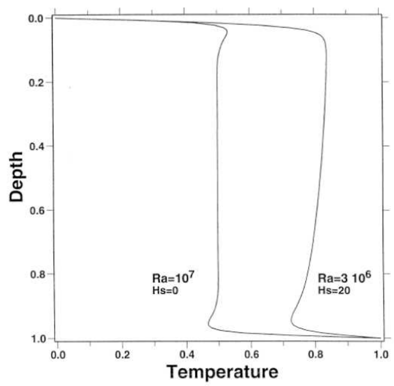

Figure 8: The non-monotonic average temperature profile found by Sotin & Labrosse [30].

$\mathrm{Nu} \leq c\mathrm{Ra}^{1/3}(\log \mathrm{Ra})^{2/3}$ [6]. More recently, Doering, Otto & Reznikoff [10] further refined the rigorous bound to $(\mathrm{Ra}\log\mathrm{Ra})^{1/3}$ by applying the background background method with a non-monotone background function. A numerical computation of the optimal background profile (which turns out to be non-monotone), suggests a simple $\mathrm{Ra}^{1/3}$ scaling for the Nu bound [14].

The need for a non-monotonic background function for the temperature was anticipated by the observations of Sotin & Labrosse [30], who carried out a detailed numerical simulation of infinite Prandtl number convection. In their simulations[2], it was apparent that cold fluid, after having decended close to the base of the convection cell, 'pools' above the thermal boundary layer due to heating of the lower boundary. This results in a small depression of mean temperature between the edge of the lower boundary layer and the well-mixed central region. The same phenomenon occurs with hot fluid just below the upper boundary layer. This 'lid' on the boundary layer stabilizes it, allowing it to grow nearly to $\mathrm{Ra}^{-1/3}$ as conjectured by the marginal stability argument given in §3.2. The resulting time-averaged temperature profiles resemble those shown in Figure 4.3.

4.4 Bounds on convection in porous media

One can also investigate convection in a porous medium, where the fluid flow is governed by Darcy's law

$$\nabla p = -\mathbf{u} + \mathrm{Ra}_D T\hat{\mathbf{z}}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{54}$$

with the Darcy-Rayleigh number

$$\mathrm{Ra}_D = \frac{g\alpha\Delta T h K}{\nu\kappa}. \tag{55}$$

K here represents the *permeability* of the medium (the square of the lengthscale of the pores, essentially a measure of the mobility of fluid within the porous matrix). In this problem, at least, there is much less controversy over the scaling of the Nusselt number for large Ra_D. Analogous arguments to those used to get the $\mathrm{Ra}^{1/2}$ and $\mathrm{Ra}^{1/3}$ scalings for

[2]viewable online at the time of writing at http://www.ipgp.jussieu.fr/∼labrosse/movies.html

open fluids both suggest that $\mathrm{Nu} \propto \mathrm{Ra}_D^1$ as $\mathrm{Ra}_D \to \infty$. This statement is further supported by an application of the background method, such as has been performed by Doering & Constantin [8].

It should be noted, however, that the scaling $\mathrm{Nu} \propto \mathrm{Ra}_D$ is rarely observed in experiments. Indeed, it is difficult to extract a single scaling law from the experimental data, as compiled by Lister [17]. Part of the problem lies in attempting to define a global Darcy-Rayleigh number on what is inevitably a heterogeneous medium. Furthermore, Darcy's law itself can be called into question, depending upon the physical parameter used to increase Ra_D. See also Otero *et al.* [24] for a detailed discussion of the analysis of convection in a porous medium.

5 Open problems and challenges

We conclude with a small subset of the many open problems and challenging aspects not yet fully explored in this interesting field.

Given the numerous inconsistent experimental results for the scaling of Nu at high Ra, one would very much like to know the ultimate state of Rayleigh-Bénard convection. Does a single ultimate state exist, or are different states reached, depending on the specifiy geometry or region of parameter space, as suggested by Grossmann & Lohse [12]?

Bounds on Nu derived via the background method do not yet include details of high-Ra dependence of Nu on the Prandtl number Pr. Is it possible to obtain sharp, uniform bounds on the Nusselt number when considering both dimensionless parameters?

Several generalisations of simple Rayleigh-Bénard convection are of interest to geophysicists, in the context of thermal transport in the mantle, or in the oceans and atmosphere. We could ask about the comparison between fixed temperature and fixed heat flux boundary conditions (see [25]), or examine the effect of adding rotation in the form of a Coriolis force (see [7], for example).

Finally, problems involving free boundaries present an interesting variation on convection problems, largely neglected in the current literature (except of course for the original closed-form linear solution of Rayleigh [28]). Not only would stress-free boundary conditions[3] alter the analysis leading to upper bounds, but more Marangoni effects, due to the variation of surface tension with temperature, could present some interesting results. Indeed, this would bring us back to our starting point, with the observations of Bénard [2].

A Derivation of integral identities for Nu

Integrating equation (15) over the domain V, we obtain

$$\frac{d}{dt}\int_V T\,dV + \int_{\partial V} T\mathbf{u}\cdot\mathbf{dS} = \int_{\partial V} \nabla T\cdot\mathbf{dS}, \tag{56}$$

[3] [23] has conjectured, based on a numerical implementation of the background method, that $\mathrm{Nu} \lesssim \mathrm{Ra}^{5/12}$ for two-dimensional convection with stress-free boundaries

where we have used the fact that $\nabla \cdot \mathbf{u} = 0$. After applying our boundary conditions to the two surface integrals, we are left with

$$\frac{d}{dt}\int_V T\,dV = \int_{z=1} \frac{\partial T}{\partial z}\,dS - \int_{z=0} \frac{\partial T}{\partial z}\,dS. \tag{57}$$

Similarly, after multiplying equation (15) by T and integrating over V, we obtain

$$\begin{aligned} \frac{d}{dt}\int_V \tfrac{1}{2}T^2\,dV + \int_{\partial V} \tfrac{1}{2}T^2\mathbf{u}\cdot\mathbf{dS} &= \int_{\partial V} T\nabla T\cdot\mathbf{dS} - \int_V |\nabla T|^2\,dV & (58)\\ \Rightarrow \frac{d}{dt}\int_V \tfrac{1}{2}T^2\,dV &= \int_{z=0} -\frac{\partial T}{\partial z}\,dS - \int_V |\nabla T|^2\,dV. & (59)\end{aligned}$$

Finally, multiplying equation (15) by z and integrating over V, we obtain

$$\begin{aligned} \frac{d}{dt}\int_V zT\,dV + \int_{\partial V} zT\mathbf{u}\cdot\mathbf{dS} - \int_V wT\,dV &= \int_{\partial V} z\nabla T\cdot\mathbf{dS} - \int_V \frac{\partial T}{\partial z}\,dV & (60)\\ \Rightarrow \frac{d}{dt}\int_V zT\,dV - \int_V wT\,dV &= \int_{z=1} \frac{\partial T}{\partial z}\,dS + \int_{z=0} dS. & (61)\end{aligned}$$

We now put (57), (59) and (61) together to find

$$\frac{d}{dt}\int_V (T - \tfrac{1}{2}T^2 - zT)\,dV + \int_V wT\,dV = \int_V |\nabla T|^2\,dV - \int_{z=0} dS. \tag{62}$$

We define the time average operator $\bar{\cdot}$ by

$$\overline{f} = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_{t=0}^{t=\mathcal{T}} f(t)\,dt \tag{63}$$

and note that, if $f(t)$ is bounded, then $\overline{\frac{df}{dt}}$ vanishes[4]. Assuming that T remains bounded throughout the domain, the time average of (62) then yields

$$\begin{aligned} \overline{\int_V wT\,dV} &= \overline{\int_V |\nabla T|^2\,dV} - \overline{\int_{z=0} dS} & (64)\\ \Rightarrow \langle wT\rangle &= \left\langle |\nabla T|^2 \right\rangle - 1. & (65)\end{aligned}$$

Furthermore, if we integrate equation (23) over the domain, we find

$$\frac{1}{\mathrm{Pr}}\frac{d}{dt}\int_V \tfrac{1}{2}u^2\,dV = -\int_V |\nabla \mathbf{u}|^2\,dV + \mathrm{Ra}\int_V Tw\,dV \tag{66}$$

and so

$$0 = -\left\langle |\nabla \mathbf{u}|^2 \right\rangle + \mathrm{Ra}\,\langle Tw\rangle\,. \tag{67}$$

[4]If, say, $|f(t)| \le C < \infty\ \forall t$ then $\frac{1}{\mathcal{T}}\left|\int_0^{\mathcal{T}} \frac{df}{dt}(t)\,dt\right| = \frac{1}{\mathcal{T}}\left|f(\mathcal{T}) - f(0)\right| \le \frac{2C}{\mathcal{T}} \to 0$ as $\mathcal{T}\to\infty$.

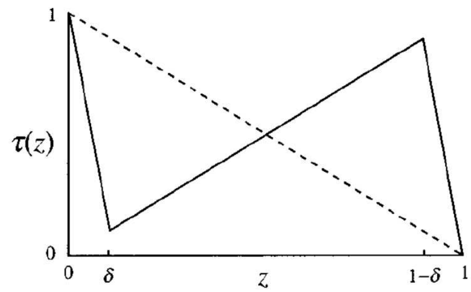

Figure 9: The piecewise linear background function (71) used in the derivation of an upper bound for Nu.

B A sample application of the background method

In this appendix, we give an example of the application of the background method to a simple case, in order to obtain a relatively crude upper bound for Nu. As in the main body of the notes (see §4.1), we write the temperature field in the form

$$T(\mathbf{x},t) = \tau(z) + \theta(\mathbf{x},t) \tag{68}$$

and consider the magnitude of the Nusselt number in the form

$$\mathrm{Nu} = \int_0^1 [\tau'(z)]^2 \mathrm{d}z - Q\{\theta, \mathbf{u}\}, \tag{69}$$

where the quadratic form Q is given by

$$Q\{\theta,\mathbf{u}\} = \left\langle |\nabla\theta|^2 + \frac{2}{\mathrm{Ra}}|\nabla\mathbf{u}|^2 + 2(\tau' - 1)w\theta \right\rangle. \tag{70}$$

We would now like to find a lower bound for Q, giving us an upper bound for Nu. Each of these bounds will depend upon the form of the background function $\tau(z)$. Whilst the first two terms in Q are positive definite, we can say very little about the last. Using our freedom to choose the background temperature field, we use a piecewise linear background $\tau(z)$ illustrated in Figure 9, and defined mathematically by

$$\tau(z) = \begin{cases} 1 - \left(\frac{1-\delta}{\delta}\right) z & \text{for } 0 < z < \delta, \\ z & \text{for } \delta < x < 1-\delta, \\ \left(\frac{1-\delta}{\delta}\right)(1-z) & \text{for } 1-\delta < z < 1, \end{cases} \tag{71}$$

which eliminates the final term in Q everywhere except within a region of width δ at each boundary. For this $\tau(z)$, we note that

$$\int_0^1 [\tau'(z)]^2 \mathrm{d}z = \frac{2}{\delta} - 3, \tag{72}$$

before concentrating our attention on bounding the quadratic form $Q\{\theta, \mathbf{u}\}$.

The absolute value of the last term in (70) is given for the background temperature field (71) by

$$\left|\int_V (\tau'(z)-1)w\theta \mathrm{d}V\right| = \frac{1}{\delta}\left|\int\int\left(\int_0^\delta w\theta \mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}x + \int\int\left(\int_{1-\delta}^1 w\theta \mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}x\right|. \tag{73}$$

Using a combination of the fundamental theorem of calculus and the Cauchy-Schwarz inequality, we can bound the temperature perturbation as follows

$$|\theta(x,y,z)| = \left|\int_0^z \frac{\partial\theta}{\partial z}(x,y,\zeta)\mathrm{d}\zeta\right| \tag{74}$$

$$\leq \left(\int_0^z \mathrm{d}\zeta\right)^{1/2}\left(\int_0^z\left(\frac{\partial\theta}{\partial z}\right)^2\mathrm{d}\zeta\right)^{1/2} \tag{75}$$

$$\leq z^{1/2}\left(\int_0^{1/2}\left(\frac{\partial\theta}{\partial z}\right)^2\mathrm{d}\zeta\right)^{1/2} \quad (\text{if} \quad z\leq 1/2). \tag{76}$$

After having found a similar bound for the vertical velocity w, and using a similar argument for $1/2 \leq z \leq 1$, we can now bound the integrals in (73) in the following way

$$\frac{1}{\delta}\left|\int\int\left(\int_0^\delta w\theta \mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}x\right| \leq \frac{\delta}{2}\int\int\left\{\left(\int_0^{1/2}\left(\frac{\partial\theta}{\partial z}\right)^2\mathrm{d}\zeta\right)^{1/2}\left(\int_0^{1/2}\left(\frac{\partial w}{\partial z}\right)^2\mathrm{d}\zeta\right)^{1/2}\right\}\mathrm{d}x\mathrm{d}y \tag{77}$$

$$\leq \frac{\delta^2\mathrm{Ra}}{16}\int_{z\leq 1/2}\left(\frac{\partial\theta}{\partial z}\right)^2\mathrm{d}V + \frac{1}{\mathrm{Ra}}\int_{z\leq 1/2}\left(\frac{\partial w}{\partial z}\right)^2\mathrm{d}V. \tag{78}$$

In obtaining the above bound, we have used the inequality $2ab \leq a^2+b^2$, with a and b carefully chosen so that the velocity gradient term will have the same prefactor (2/Ra) as in the expression for $Q\{\theta,\mathbf{u}\}$ (70). After combining the contributions from the boundary layers near 0 and 1, we can write

$$\left|\langle 2(\tau'(z)-1)w\theta\rangle\right| \leq \frac{\delta^2\mathrm{Ra}}{8}\int_V\left(\frac{\partial\theta}{\partial z}\right)^2\mathrm{d}V + \frac{2}{\mathrm{Ra}}\int_V\left(\frac{\partial w}{\partial z}\right)^2\mathrm{d}V, \tag{79}$$

therefore we can bound the quadratic form Q from below as follows

$$Q\{\theta,\mathrm{u}\} = \left\langle |\nabla\theta|^2 + \frac{2}{\mathrm{Ra}}|\nabla \mathrm{u}|^2 + 2(\tau'-1)w\theta\right\rangle \tag{80}$$

$$\geq \left\langle\left(\frac{\partial\theta}{\partial z}\right)^2 + \frac{2}{\mathrm{Ra}}\left(\frac{\partial w}{\partial z}\right)^2 - \frac{\delta\mathrm{Ra}}{8}\left(\frac{\partial\theta}{\partial z}\right)^2 - \frac{2}{\mathrm{Ra}}\left(\frac{\partial w}{\partial z}\right)^2\right\rangle \tag{81}$$

$$= \left(1-\frac{\delta^2\mathrm{Ra}}{8}\right)\left\langle\left(\frac{\partial\theta}{\partial z}\right)^2\right\rangle. \tag{82}$$

If we use our freedom to choose the background function so that $\delta = 2\sqrt{2/\mathrm{Ra}}$, we can guarantee that $Q \geq 0$ for all θ and u. Substituting this information into equations (71) and (72), we find that

$$\mathrm{Nu} \leq \sqrt{\frac{\mathrm{Ra}}{2}} - 3. \tag{83}$$

For consistency, we require that $\delta \leq 1/2$, as the background function must be single-valued. The above analysis therefore only holds for $\mathrm{Ra} \geq 32$.

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