

GFD 2013 Lecture 3: Hydraulic theories

Paul Linden; notes by Ton van den Bremer & Dhruv Balwada

June 19, 2013

1 Introduction

This lecture explores the use of hydraulic theories, theories that originated in the study of the behaviour of water in pipes or tube after the Greek words *hydro* for water and *aulis* for tube or pipe, to study gravity currents. It is exactly the motion of an air bubble in a pipe, illustrated in figure 1, that inspired the work by Benjamin [1] on gravity currents using the conservation laws of mass, momentum and energy that will be explored in this lecture.

2 Heavy current

Figure 2 gives a schematic representation of a dense flow with density ρ_L , the gravity current, protruding at the bottom of a two-dimensional horizontal channel filled with a lighter stagnant fluid of density ρ_U . We consider the problem in the reference frame in which the gravity current, which moves to the right relative to the channel with velocity U , is stationary. The depth of the ‘downstream’ (BE) end of the gravity current is h with the upper fluid moving at velocity u_U to the left (in the reference frame of the current), and the total depth of the channel is H . We assume a unit width throughout. In the reference frame of the current, there is a stagnation point at O .

Conservation of mass flux across the control volume $BCDE$ gives:

$$UH = u_U(H - h). \quad (1)$$

We further assume that the flow is horizontal, uniform across depth and that, accordingly, the pressure is hydrostatic. The vertical pressure distributions across BE and CD are therefore respectively given by:

$$BE : \quad p(z) = \begin{cases} p_B - g\rho_L z & 0 \leq z \leq h, \\ p_B - g\rho_L h - g\rho_U(z - h) & h \leq z \leq H, \end{cases} \quad (2)$$

$$CD : \quad p(z) = p_C - g\rho_U z, 0 \leq z \leq H \quad (3)$$

where z is measured from the bottom of the channel upwards, and gravity g acts in the negative z direction. Applying conservation of momentum across the control volume $BCDE$ ignoring the effects of the channel walls on the fluid through viscosity corresponds to:

$$\int_{z_C}^{z_D} (p + \rho u^2) dz = \int_{z_B}^{z_E} (p + \rho u^2) dz, \quad (4)$$

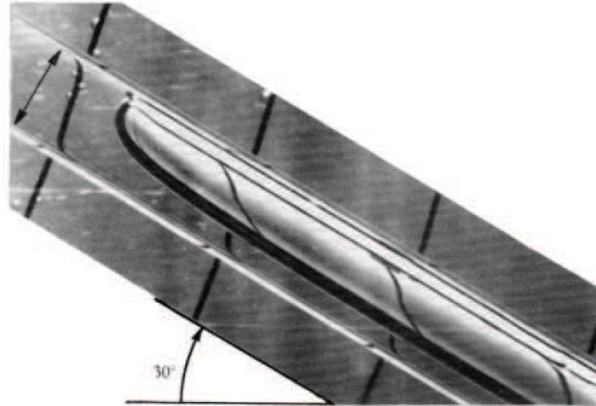


Figure 1: A bubble in an inclined closed tube from [3].

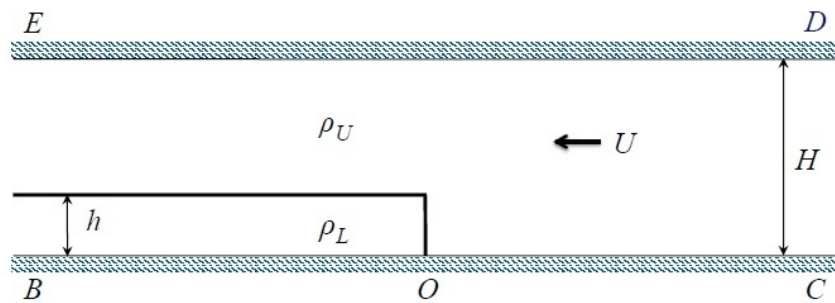


Figure 2: A schematic diagram of an idealized dense gravity current in a reference frame in which the current is at rest. In the laboratory reference frame, the current is propagating with constant speed U into an ambient fluid at rest. It is assumed that all the fluid within the current has the same speed.

which, in turn by simple rearranging and evaluation of the integrals, can be shown to correspond to:

$$p_B H + \frac{1}{2}g(\rho_L - \rho_U)h^2 - g(\rho_L - \rho_U)Hh + \rho_U u_U^2(H - h) = p_C H + \rho_U U^2 H. \quad (5)$$

Equation (5) does not define the relationship between the pressures p_B and p_C that is required to close the problem. We apply Bernoulli along the lower boundary of the channel, which is evidently a streamline, from C to O and from O to B and obtain:

$$p_O = p_C + \frac{1}{2}\rho_U U^2 = p_B. \quad (6)$$

Substituting for u_U in terms of U from conservation of mass (1) and for the pressure difference $p_C - p_B$ from (6), equation (5) can be rewritten in the following form:

$$\frac{U^2}{g(1 - \gamma)H} = \frac{1}{\gamma}f\left(\frac{h}{H}\right), \quad (7)$$

where $0 \leq \gamma = \rho_U/\rho_L, \leq 1$ and $f(h/H)$ is given by:

$$f\left(\frac{h}{H}\right) = f(\hat{h}) = \frac{\hat{h}(2 - \hat{h})(1 - \hat{h})}{1 + \hat{h}}, \quad (8)$$

where $\hat{h} = h/H$. In the Boussinesq case, in which density difference are assumed to be small $\gamma \rightarrow 1$, the Froude number and the non-dimensional volume flux are then respectively given by:

$$F_H = \frac{U}{\sqrt{g'H}} = \sqrt{f(\hat{h})}, \quad \frac{Q}{\sqrt{g'H^3}} = \frac{Uh}{\sqrt{g'H^3}} = \hat{h}\sqrt{f(\hat{h})}. \quad (9)$$

Figure 3 shows both the Froude number F_H and the non-dimensional volume flux in (9) as a function of h/H . To find h itself, a further condition is needed.

2.1 The energy conserving case

Following Benjamin's [1] approach, we assume there are no energy losses, so that we can apply Bernoulli from E to D and obtain:

$$p_E + \frac{1}{2}\rho_U u_U^2 = p_D + \frac{1}{2}\rho_U U^2 \quad (10)$$

Combining (10) with conservation of mass (1), assuming there is a hydrostatic relationship between p_E and p_B and p_D and p_C , respectively, and that the relationship (6) between p_B and p_C still holds, we obtain:

$$\frac{U^2}{g(1 - \gamma)H} = \frac{2}{\gamma} \frac{h(H - h)^2}{H^3}. \quad (11)$$

Equating (11) and the analogous expression derived without assuming zero energy losses (7) gives:

$$2\hat{h}(1 - \hat{h})^2 = \frac{\hat{h}(2 - \hat{h})(1 - \hat{h})}{1 + \hat{h}}, \quad (12)$$

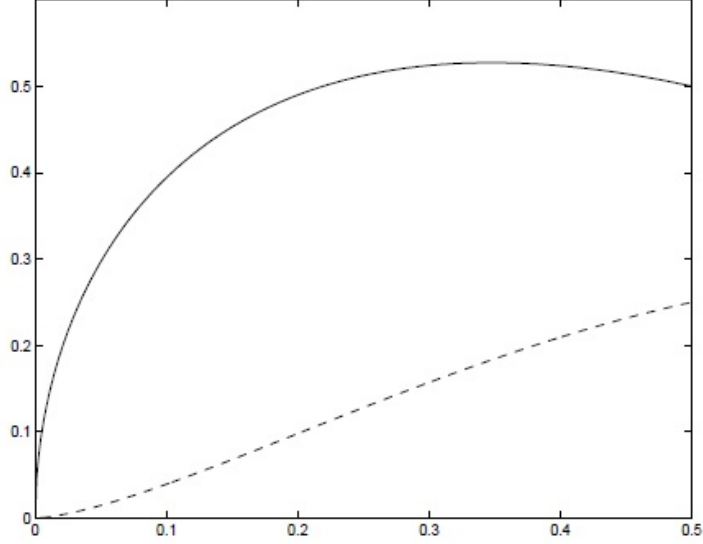


Figure 3: The Froude number F_H (continuous line) and the dimensionless volume flux $Q/\sqrt{g'H^3}$ (dashed line) as a function of the dimensionless gravity current depth $\hat{h} = h/H$.

where $\hat{h} = h/H$ as before. Equation (12) has three solutions: $\hat{h} = 0$, for which there is no gravity current, $\hat{h} = 1/2$ and $\hat{h} = 1$, for which the gravity current fills the entire channel. The only physically relevant solution is $h = H/2$ and, by substituting into (9) this can be shown to correspond to $F_H = 1/2$.

Now consider the Froude numbers based on the gravity current depth h for the energy conserving case $h = H/2$ and take the Boussinesq limit $\gamma \rightarrow 1$. From (11) we obtain:

$$F_h = \begin{cases} \frac{U}{\sqrt{g'h}} = \frac{U}{\sqrt{g'H}} \sqrt{\frac{H}{h}} = \frac{1}{\sqrt{2}} < 1 & \text{'upstream' (sub-critical),} \\ \frac{u_U}{\sqrt{g'h}} = \frac{2U}{\sqrt{g'h}} = \sqrt{2} > 1 & \text{'downstream' (super-critical),} \end{cases} \quad (13)$$

where 'upstream' and 'downstream' refer to the reference frame in which the current is stationary and we have used $u_U = 2U$ from mass conservation for $H = 2h$. In the downstream region of the control volume (the left-hand side) the flow is thus super-critical and waves cannot propagate faster than the speed of the gravity current. The sub-critical upstream region (the right-hand side) is left undisturbed until the arrival of the gravity current.

2.2 Energy losses

The energy flux \dot{E} across a vertical plane is defined as follows:

$$\dot{E} = \int_0^H (p + \frac{1}{2}\rho u^2 + g\rho)udz. \quad (14)$$

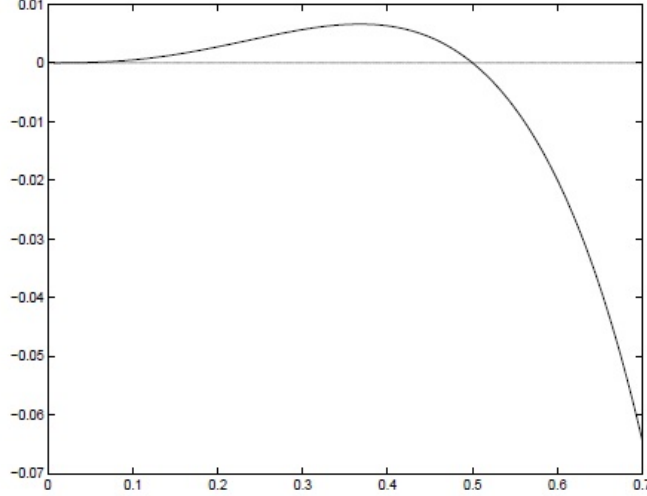


Figure 4: The dimensionless net energy loss flux $\Delta\dot{E}/gH^2U(\rho_L - \rho_U)$ as a function of the dimensionless gravity current depth $\hat{h} = h/H$.

Expressions for the energy fluxes at BE and CD are given by, respectively:

$$\dot{E}_{BE} = \left(p_B - g(\rho_L - \rho_U)h + \frac{1}{2}\rho_U u_U^2 \right) u_U H, \quad (15)$$

$$\dot{E}_{CD} = \left(p_C + \frac{1}{2}\rho_U U^2 \right) UH. \quad (16)$$

The energy loss flux is defined as $\Delta\dot{E} = \dot{E}_{CD} - \dot{E}_{BE}$. After substituting in for u_U from mass conservation (1) and momentum conservation (5) relating the pressures p_B and p_C by (6), the non-dimensional energy loss flux is defined by:

$$\frac{\Delta\dot{E}}{gH^2U(\rho_L - \rho_U)} = \frac{\hat{h} - \frac{1}{2}f(\hat{h})\frac{1}{(1-\hat{h})^2}}{1 - \hat{h}}. \quad (17)$$

Figure 4, which shows the non-dimensional energy loss as a function of h/H , reveals there are only two solutions for which the energy losses described by (17) are zero: $h/H = 0$ and $h/H = 1/2$. It is evident from this figure that energy losses are only positive for $0 < h/H < 1/2$. For $h/H > 1/2$ energy needs to be input into the system. It is therefore impossible to maintain a gravity current with $h/H > 1/2$ unless the channel is tilted or energy is input into the flow in another way. In practical terms, such considerations are relevant for flushing pipeline applications, where it may hence not be possible to flush one liquid from the pipe entirely using another without tilting the pipe.

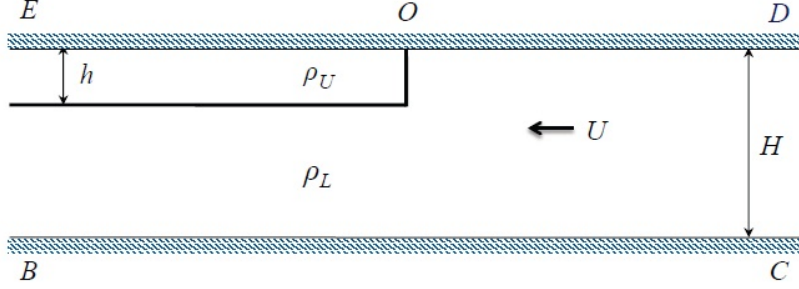


Figure 5: A schematic diagram of an idealized light gravity current in a reference frame in which the current is at rest. In the laboratory reference frame, the current is propagating with constant speed U into an ambient fluid at rest and it is assumed that all the fluid within the current has this same speed.

3 Light current

We now consider a light gravity current protruding into a heavy fluid at the top, as illustrated in figure 5. From conservation of mass we obtain:

$$u_L(H - h) = UH, \quad (18)$$

where u_L now denotes the horizontal velocity of the lower fluid at location BE in the reference frame of the gravity current. Assuming a hydrostatic pressure distribution, the pressure difference between BE and CD is given by:

$$\Delta p(z) = p_E - p_D + \begin{cases} (\rho_U - \rho_L)gz & 0 \leq z \leq h, \\ (\rho_U - \rho_L)gh & h \leq z \leq H, \end{cases} \quad (19)$$

where z is measured from the top of the channel down in the direction of gravitational acceleration g . Equating momentum fluxes across BE and CD gives:

$$\int_0^H \Delta p(z) + \rho_L u_L^2(H - h) - \rho_L U^2 = 0. \quad (20)$$

Integrating the pressure difference distribution (19), substituting for u_L from mass conservation (18) gives:

$$\frac{U^2}{g(1 - \gamma)H} = f(\hat{h}), \quad (21)$$

where $\hat{h} = h/H$ and $f(\hat{h})$ is given by (8) as for the heavy gravity current. Comparison between (7) for a heavy current and (21) for a light current reveals a factor of $1/\gamma$. A heavy current moves faster than a light current ($\gamma < 1$).

4 Flow at the stagnation point

To understand the flow at the stagnation point and to find the local angle between the front and the edge of the current α (see figure 6), we solve for the potential flow field. We assume

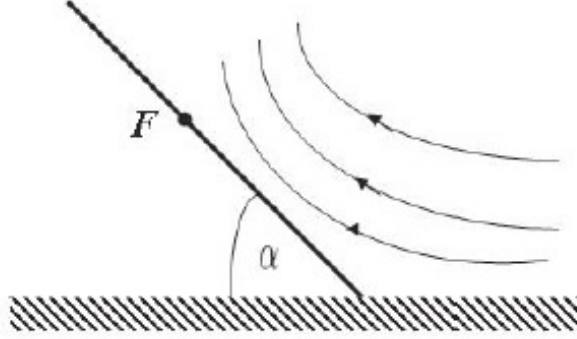


Figure 6: The flow in a corner representative of potential flow at the gravity current front.

uniform flow in the direction of the real axis of the complex domain w with corresponding complex potential $\Omega(w) = cw$, where c is a constant and for which the horizontal and vertical velocity are respectively given by $\text{Re}[d\Omega(w)/dw] = c$ and $-\text{Im}[d\Omega(w)/dw] = 0$. It is evident then that $\Omega(w)$ satisfies Laplace in the infinite half-space domain. We use a two-dimensional conformal mapping, to map to the space $z = r \exp(i\theta)$ in which the flow of the ambient fluid is confined by the gravity current (cf. figure 6 showing the flow field in z -space):

$$w = z^{\frac{\pi}{\pi-\alpha}} = r^{\frac{\pi}{\pi-\alpha}} e^{i\frac{\pi}{\pi-\alpha}\theta}. \quad (22)$$

The complex potential in z -coordinates is then given by:

$$\Omega(z) = \phi + i\psi = cz^{\frac{\pi}{\pi-\alpha}}. \quad (23)$$

The horizontal velocity can now be expressed as a function of the radial coordinate r :

$$q^2 = \frac{d\Omega}{dz} \frac{d\Omega^*}{dz} = c^2 \left(\frac{\pi - \alpha}{\phi} \right)^2 r^{2\frac{\alpha}{\pi-\alpha}}, \quad (24)$$

where Ω^* is the complex conjugate of Ω . Applying Bernoulli along a streamline that takes a fluid particle from to the far right of the stagnation point elevating it and accelerating it to velocity q once it comes in the vicinity of the gravity current front, we obtain the equality:

$$\frac{1}{2}\rho q^2 = g\rho \sin(\alpha). \quad (25)$$

Substituting for q^2 from (24) into (25) and matching powers of r , we require $\alpha = \pi/3$ for the solution to Laplace to satisfy Bernoulli. For a current that flows down a slope of angle θ , $\sin(\alpha)$ in (25) is simply replaced by $\sin(\alpha + \theta)$. A maximum flow speed q then corresponds to $\sin(\alpha + \theta) = 1$ and thus to $\theta = \pi/2 - \alpha = \pi/6$.

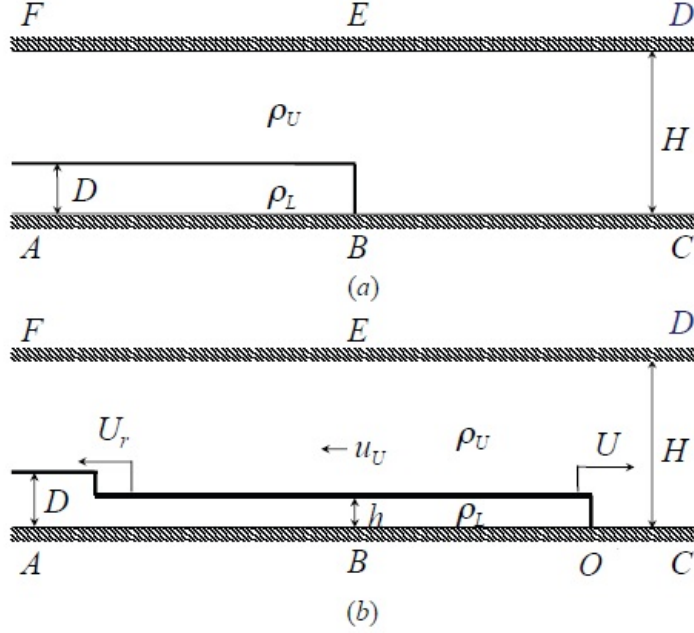


Figure 7: Schematic of a partial-depth lock release in a channel before release (a) and after release (b). Velocities are in the reference frame of the stationary channel.

5 Partial depth lock

Having only considered full-depth releases in a closed channel in the previous sections, figure 7 outlines what happens in the case of a partial depth lock release. In addition to a front travelling to the right with velocity U and depth h , there is a disturbance moving to the left with velocity U_r (both measured relative to the stationary closed channel) associated with a jump in fluid depth of h to D . At the far left hand side of the fluid (AF) the fluid is at rest (relative to the closed channel). Above the current of ρ_L , that is, to the right of the disturbance and to the left of the front, the upper fluid of density ρ_U moves to the left with speed u_U . Conservation of mass across the front is most easily considered in the reference frame in which the front is stationary:

$$UH = (U + u_U)(H - h) \rightarrow u_U = \frac{Uh}{H - h}. \quad (26)$$

Conservation of mass across the disturbance in the reference frame of the channel, gives a zero flux out on the left hand side and a positive flux out on the right hand side of magnitude Uh associated with the current moving to the right. This net outflux is balanced by a loss of total volume contained in the control volume of $U_r(D - h)$:

$$Uh = U_r(D - h) \rightarrow U_r = \frac{Uh}{D - h}. \quad (27)$$

We return to the reference frame of the stationary channel and consider a control volume just outside of both the left-travelling disturbance and the right-travelling gravity current

front. The distribution of the pressure difference between the left-hand side (AF) and the right-hand side is (CD):

$$\Delta p(z) = p_F - p_D \begin{cases} 0 & \leq z \leq H - D, \\ (\rho_L - \rho_U)g(z - (H - D)) & H - D \leq z \leq H, \end{cases} \quad (28)$$

where z is measured from the top of the channel down. Integrating (28) with respect to z across the depth of the channel gives net force applied to the control volume $ACDF$:

$$\dot{M} = \frac{1}{2}g(\rho_L - \rho_U)D^2 + (p_F - p_D)H. \quad (29)$$

Having considered the forces applied to the control volume, we consider the change in momentum that results. In the stationary reference frame we have for the lower fluid:

$$\dot{M}_L = \rho_L(U + U_r)Uh \quad (30)$$

where the momentum per unit mass is U , and the rate of change of the volume that has this momentum is $dV/dt = hdL/dt = h(U + U_r)$. Here, the length scale L denotes the length of the gravity current. A unit width is assumed throughout this lecture. The upper fluid has momentum per unit mass $-u_U$:

$$\dot{M}_U = -\rho_U(U + U_r)(H - h)u_U, \quad (31)$$

where the rate of change of the volume that has momentum $-u_U$ is $dV/dt = (H - h)dL/dt = (H - h)(U + U_r)$. Here, again the length scale L denotes the length of the gravity current. Equating $\dot{M} = \dot{M}_L + \dot{M}_U$, where \dot{M} is given by (29) and \dot{M}_L and \dot{M}_U by (30) and (31), respectively, gives:

$$(p_D - p_F)H = (\rho_L - \rho_U)\left[U^2 \frac{Dh}{D - h} + \frac{1}{2}gD^2\right]. \quad (32)$$

What remains to be found is the pressure difference $p_D - p_F$. Since with the two different velocities, U and U_r , a reference frame can no longer be found in which the problem is steady, we apply unsteady Bernoulli between D and F to find the pressure difference between these two points:

$$p_F + \rho_U \frac{\partial \phi_U}{\partial t} \Big|_F = p_D + \rho_U \frac{\partial \phi_U}{\partial t} \Big|_D, \quad (33)$$

where ϕ is the velocity potential. For (33) to hold, energy conservation in the top layer is assumed. Letting x_f denote the position of the front, x_r the position of the left travelling disturbance, the horizontal velocity in the top layer is given by:

$$u = \begin{cases} 0 & \text{for } x < x_r, \\ -u_U & \text{for } x_r < x < x_f, \\ 0 & \text{for } x > x_f. \end{cases} \quad (34)$$

Integrating (34) with respect to x gives the velocity potential:

$$\phi_U = \begin{cases} 0 & \text{for } x < x_r, \\ -u_U(x - x_r) & \text{for } x_r < x < x_f, \\ -u_U(x_f - x_r) & \text{for } x > x_f, \end{cases} \quad (35)$$

which is defined up to an arbitrary constant but does not include an arbitrary dependence on z , as the no-flow boundary condition requires $\partial\phi/\partial z = 0$ along the boundary of the channel. The pressure difference $p_D - p_F$ is then equal to:

$$p_D - p_F = \rho_U u_U (\dot{x}_f - \dot{x}_r) = \rho_U u_U (U + U_r). \quad (36)$$

Combining conservation of mass (26-27), momentum (32) and substituting for the pressure difference from (36) gives after some manipulation:

$$\frac{U^2}{gH} = \frac{(\rho_L - \rho_U)D(D-h)(H-h)}{2hH(\rho_L(H-h) + \rho_U h)}. \quad (37)$$

5.1 The energy conserving case

Finally, we assume energy is conserved to close the problem. There are no fluxes of energy into or out of the control volume $ACDF$. There is a decrease in potential energy associated with the loss in elevation of the interface position of the undisturbed fluid due to the disturbance travelling to the left and an increase in the potential energy associated with the increase in elevation due to the gravity current travelling to the right:

$$\dot{E}_P = -\frac{1}{2}g(\rho_L - \rho_U)U_r(D^2 - h^2) + \frac{1}{2}(\rho_L - \rho_U)U h^2, \quad (38)$$

where $(1/2)g(\rho_L - \rho_U)(D^2 - h^2)$ and $(1/2)g(\rho_L - \rho_U)h^2$ are the potential energy per unit length associated with the disturbance and the gravity current, respectively. The rates of change of the respective length scales associated with these changes in potential energy are $dL/dt = -U_r$ for the disturbance and $dL/dt = U$ for the gravity current. The change in potential energy (38) is matched by an increase in kinetic energy of the current and the disturbance:

$$\dot{E}_K = \frac{1}{2}\rho_L U^2 (U + U_r)h + \frac{1}{2}\rho_U u_U^2 (H - h). \quad (39)$$

Equating $\dot{E}_P = \dot{E}_K$ from (38) and (39) and thereby assuming conservation of energy gives after some rearranging:

$$\frac{U^2}{gH} = \frac{(\rho_L - \rho_U)(D-h)(H-h)}{H(\rho_L(H-h) + \rho_U h)}. \quad (40)$$

Comparison of (40) and (37) reveals that the only solution that conserves energy thus has $h = D/2$ consistent with the analogous result for full-depth lock releases. Figure 8 compares this finding to the data providing support for the assumption of zero energy losses.

In the Boussinesq limit $\gamma \rightarrow 1$, (40) setting $h = D/2$ reduces to:

$$\frac{U^2}{g'H} = \frac{D(2H-D)}{4H^2}, \quad (41)$$

or in terms of a Froude number:

$$F_D = \frac{U}{\sqrt{g'D}} = \sqrt{\frac{1}{2} - \frac{1}{4} \frac{D}{H}}, \quad (42)$$

which is compared to experimental evidence in 9.

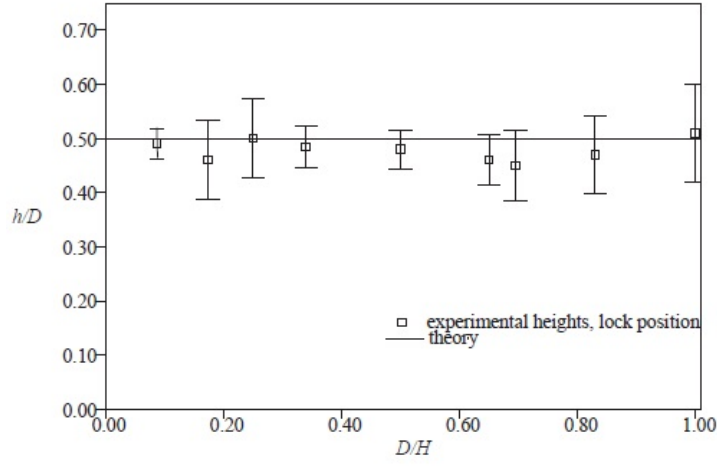


Figure 8: Comparison of measurements with the theoretical prediction (solid line) of the depth of the gravity current for partial-depth lock releases from [2].

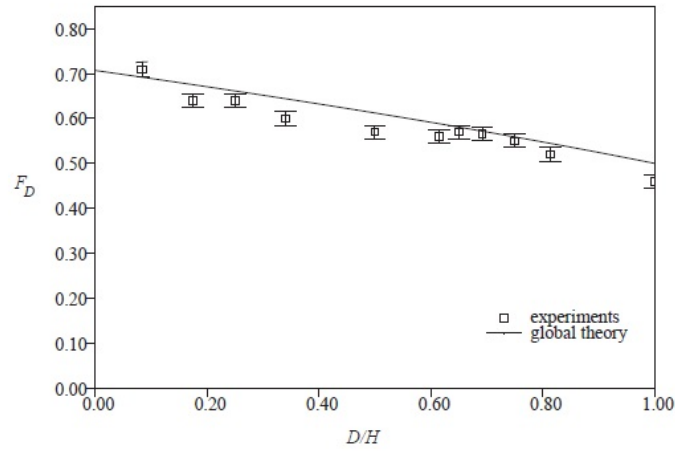


Figure 9: Comparison of measurements with the theoretical prediction (solid line) of the gravity current speeds, expressed non-dimensionally as Froude numbers based on the depth of the lock D from [2].

5.2 Deep and shallow locks

In the limit of a full depth lock $D = H$, (41) reduces to $F_H = 1/2$, as for a full depth lock. In the shallow lock limit, we can write:

$$\frac{U^2}{g'h} = 1 - \frac{h}{H} \rightarrow 1 \quad \text{as} \quad \frac{h}{H} \rightarrow 0. \quad (43)$$

5.3 Interfacial long waves

Finally, we compare the speed of the gravity current in a partial depth lock release to the phase speed of interfacial waves:

$$c_{\pm} = \frac{u_U h + u_L(H - h)}{H} \pm \frac{1}{H} \sqrt{h(H - h)(g'H - (u_U - u_L)^2)}, \quad (44)$$

which can be obtained from solving the linear long wave equation in a two-layer fluid, where the upper layer of depth $H - h$ has velocity u_U in the negative x -direction and the lower layer of depth h has velocity u_L in the positive x -direction. Normalisation of (44) by the solution for the horizontal front speed of the partial-depth lock release problem (41) and setting $U_L = U$ and $U_U = -Uh/(H - h)$ from mass conservation gives:

$$\frac{c_{\pm}}{U} = \frac{2(H - D)}{2H - D} \pm \sqrt{\frac{2(H - D)}{2H - D}}. \quad (45)$$

It is evident from (45) and from figure 10 that for shallow lock releases ($D/H < 0.76$) wave to the right travel faster than the current, whereas for deep enough locks the flow speed is super-critical ($c/U < 1$) even for the fast right-travelling waves. These waves are also evident in figure 11, in which they travel towards the front and accumulate there for the partial-depth release (figure 11a) and are stationary for the full-depth release (11b).

References

- [1] T. B. BENJAMIN, *Gravity currents and related phenomena*, J. Fluid Mech., 31 (1968), pp. 209 – 248.
- [2] J. O. SHIN, S. B. DALZIEL, AND P. F. LINDEN, *Gravity currents produced by lock exchange*, J. Fluid Mech., 521 (2004), pp. 1–34.
- [3] E. E. ZUKOSKI, *Influence of viscosity, surface tension and inclination angle on the motion of long bubbles in closed tubes*, J. Fluid Mech., 25 (1966), pp. 821–840.

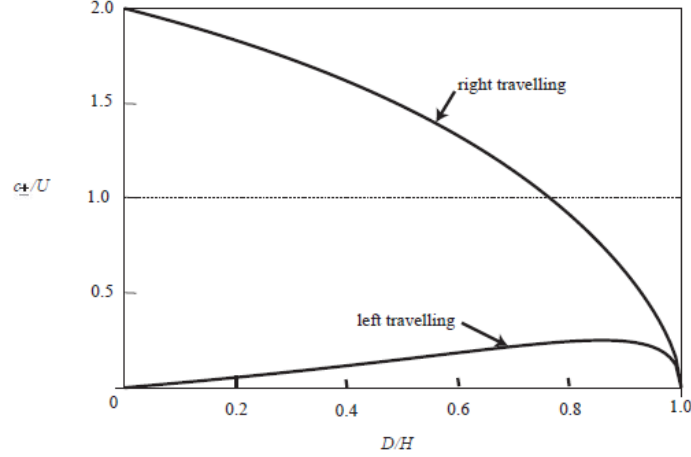


Figure 10: The speeds c_{\pm}/U of waves on the top of the current as a function of the fractional depth D/H of the release. Waves travelling to the left are always slower than the current, while waves travelling to the right are faster for $D/H < 0.76$.

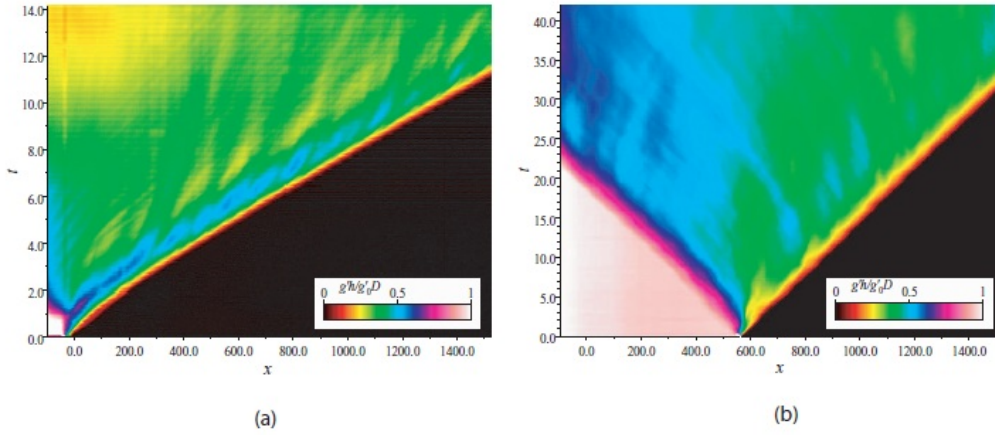


Figure 11: The buoyancy $g'h$ shown by the false colour plotted on an $x-t$ plot for a partial-release $D/H = 0.21$ (a) and a full-depth release $D/H = 1$ in (b). The intensities are normalized by the initial buoyancy $g'_0 h$ in the lock. The front position at any time is the location at the edge of the black region, and the constant front speed is shown by the straight line fit to this shown in (a). The blue regions behind the front are elevated values of the buoyancy indicating deep regions of the current. They travel towards the front in (a), but are almost stationary in (b).