Lecture 6: Derivation of the KdV equation for surface and internal waves

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1 Introduction

We sketch here a different derivation of the Korteweg–de Vries equation, applicable to a wider range of problems, including internal waves and waves in sheared flows in addition to surface water waves. After beginning with the Euler equations and discarding small terms, the problem is reduced to a superposition of various modes — but through dispersion, after sufficient time has passed each mode becomes isolated, so we may consider solitary wave (soliton) solutions. The problem is then treated with an asymptotic expansion in small parameters characterizing the amplitude and dispersion. To first order we recover the linear long wave solutions, and to second order we find that the amplitude evolves according to the KdV equation. The extended KdV (eKdV) equation is discussed for critical cases where the quadratic nonlinear term is small, and the lecture ends with a selection of other possible extensions.

2 Derivation for surface and internal waves: Basic Setup

In the basic state, the motion is assumed to be two-dimensional and the fluid has a density $\rho_0(z)$, a corresponding pressure $p_0(z)$ such that the background state is hydrostatic ($p_{0z}(z) = -g\rho_0$), and a horizontal shear flow $u_0(z)$ in the x-direction. In this lecture we consider only the case where the bottom is flat (h is constant). Extensions considering variable depth are possible and lead to a variable-coefficient KdV equation (see Lecture 9).

The equations of motion relative to this basic state are the Euler equations:

$$\rho_0(u_t + u_0u_x + wu_{0z}) + p_x = -\rho_0(uu_x + wu_z) - \rho(u_t + u_0u_x + wu_{0z} + uu_x + wu_z) \quad (1)$$

$$p_z + g\rho = -(\rho_0 + \rho)(w_t + u_0w_x + uw_x + ww_z) \quad (2)$$

$$g(\rho_t + u_0\rho_x) - \rho_0N^2w = -g(u\rho_x + w\rho_z) \quad (3)$$

$$u_x + w_z = 0 \quad (4)$$

Here $(u_0 + u, w)$ are the velocity components in the (x, z) directions, $\rho_0 + \rho$ is the density, $p_0 + p$ is the pressure and t is time. The equations are arranged such that for linear long waves all terms on the right hand side can be neglected (this statement is proved in Section 4). Here $N(z)$ is the buoyancy frequency, defined by

$$\rho_0 N^2 = -g\rho_{0z}. \tag{5}$$

The boundary conditions are

$$w = 0 \text{ at } z = -h, \tag{6}$$

$$p_0 + p = 0 \text{ at } z = \eta, \tag{7}$$

$$w = \eta_t + u_0\eta_x + u\eta_x \text{ at } z = \eta. \tag{8}$$

It is useful to use the vertical particle displacement ζ as the primary dependent variable. ζ is readily measured in the ocean and is related to the buoyancy frequency. The vertical particle displacement is defined by

$$\frac{D\zeta}{Dt} = \zeta_t + u_0\zeta_x + u\zeta_x + w\zeta_z = w. \tag{9}$$

The perturbed density field is given by the difference between the density at the vertical position a fluid particle originates from, and that of its current location: $\rho = \rho_0(z-\zeta) - \rho_0(z) \approx g\rho_0 N^2\zeta$ as $\zeta \to 0$, where we have assumed that as $x \to -\infty$, the density field relaxes to $\rho_0(z)$. Isopycnal surfaces (i.e. $\rho_0 + \rho =$constant) are then given by $z = z_0 + \zeta$ where z_0 is the far-field level. The physical meaning of ζ is clearest on the free surface ($z = \eta$). In terms of ζ, the kinematic boundary condition (8) becomes $\zeta = \eta$ at $z = \eta$.

3 Linear long waves

To describe internal solitary waves, we seek solutions whose horizontal length scales are much greater than h (shallow water, long waves) and whose time scales are much greater than N^{-1} (linear behavior). We now solve the Euler equations by omitting the right-hand side of equations (1-4) and utilizing the linearized free-surface boundary conditions of (7,8). Solutions are sought in the form

$$\zeta = A(x - ct)\phi(z) \tag{10}$$

Plugging this solution for ζ into the linear Euler equations gives values for the remaining dependent variables:

$$u = A\left((c-u_0)\phi\right)_z,\ p = \rho_0(c-u_0)^2 A\phi_z,\ g\rho = \rho_0 N^2 A\phi. \tag{11}$$

Here c is the linear long wave speed, and the modal functions $\phi(z)$ are defined by the boundary-value problem

$$\{\rho_0(c-u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0 \text{ for } h < z < 0 \tag{12}$$

$$\phi = 0 \text{ at } z = -h,\ (c-u_0)^2\phi_z = g\phi \text{ at } z = 0. \tag{13}$$

Equation (12) is the well-known Taylor-Goldstein equation for perturbations in stratified shear flows, here expressed in the long-wave limit.

Typically, the boundary value problem (12,13) has an infinite sequence of linear long-wave modes solutions, $\phi_n^{\pm}(z)$, $n = 0, 1, 2, \ldots$, with corresponding wave speeds, $c_n^{\pm}(z)$. Here, the superscript "$\pm$" indicates waves with $c_n^+ > u_M = \max\, u_0(z)$ and $c_n^- < u_m = \min\, u_0(z)$. We shall confine our attention to these regular modes, and consider only stable shear flows. Mathematically, this implies that we do not consider modes with complex eigenvalues c, nor modes with $c \in [u_m, u_M]$. Analogous theory can be developed for singular modes with $u_m < c < u_M$.

In general, this boundary value problem has to be solved numerically. Typically, the $n = 0$ mode denotes the surface gravity waves for which c scales with $\sqrt{gh}$, while the $n = 1, 2, 3, \ldots$ modes denote internal gravity waves for which c scales with Nh. The surface mode ϕ_0 has no extrema in the interior of the fluid and takes its maximum value at $z = 0$. The internal modes $\phi_n^{\pm}(z)$, $n = 1, 2, 3, \ldots$, have $n - 1$ extremal points in the interior of the fluid and ϕ nearly vanishes near $z = 0$ (because $c^2 \ll gh$ for internal waves, and using equation (13)).

The solution of the linearized long wave equations is given asymptotically[1] by

$$\zeta \sim \sum_{n=0}^{\infty} A_n^{\pm}(x - c_n^{\pm} t)\phi_n^{\pm}(z) \text{ as } t \to \infty. \tag{14}$$

Here the amplitudes $A_n^{\pm}$ are determined from initial conditions. Assuming that the speeds $c_n^{\pm}$ of each mode are sufficiently distinct, the modes will separate spatially for large times, so we can consider a single mode in isolation. Henceforth, we shall omit the index n and assume that the single mode has speed c, amplitude A and modal function $\phi(z)$.

4 Asymptotic expansion

Having waited sufficiently long for the modes to separate also implies that hitherto neglected nonlinear terms may begin to have an effect. The nonlinear effects are balanced by dispersion (also neglected in linear long wave theory); this balance emerges as time increases and results in the Korteweg-deVries equation for the wave amplitude.

The formal derivation of the evolution equation requires the introduction of two small parameters, ε and δ, characterizing the wave amplitude and inverse wave-length, respectively. As seen in Lecture 5, for nonlinearity to be balanced by dispersion (KdV balance), it is required that $\varepsilon = \delta^2$ (note that the leading nonlinear term is of order ϵ^2 while the leading dispersive term is of order $\epsilon\delta^2$). It was also shown that the nonlinear dynamics take place on a slow timescale T.

As in Lecture 5, we first introduce the scaled variables

$$T = \delta\varepsilon t, \;\; X = \delta(x - ct). \tag{15}$$

We then assume solutions of the form

$$\zeta = \varepsilon A(X, T)\phi(z) + \varepsilon^2 \zeta_2 + \ldots \tag{16}$$

[1]The reason why this is only valid asymptotically and not for all times is a technical issue: the modes defined do not necessarily form an orthogonal set so additional transient terms must be added to match the initial conditions in some cases.

with similar expressions analogous to 11 for the other dependent variables. Plugging these solutions into the full Euler equations results in the linear long wave solution for the modal function, $\phi(z)$ and the speed, c, at leading order. Since the modal equations (12,13) are homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. In this case, the amplitude εA is uniquely defined as the amplitude of ζ to $\mathrm{O}(\varepsilon)$ at the depth z_m.

Continuing the expansion to the next order in ε, it can be shown that this leads to the following equation for ζ_2:

$$\{\rho_0(c-u_0)^2\zeta_{2Xz}\}_z + \rho_0 N^2\zeta_{2X} = M_2, \text{ for } -h < z < 0 \tag{17}$$

and the corresponding boundary condition

$$\zeta_{2X} = 0 \text{ at } z = -h, \ \rho_0(c-u_0)^2\zeta_{2Xz} - \rho_0 g\zeta_{2X} = N_2 \text{ at } z = 0. \tag{18}$$

The inhomogeneous terms M_2, N_2 are due to nonlinearity and dispersion, and are known in terms of the first-order functions $A(X,T)$ and $\phi(z)$. They are given by

$$M_2 = 2\{\rho_0(c-u_0)\phi_z\}_z A_T + 3(\rho_0(c-u_0)^2\phi_z^2)_z AA_X - \rho_0(c-u_0)^2\phi A_{XXX}, \tag{19}$$

$$N_2 = 2\{\rho_0(c-u_0)\phi_z\}A_T + 3(\rho_0(c-u_0)^2\phi_z^2)AA_X \tag{20}$$

Equations (17,18) are identical to the equations defining the modal function (12,13), with an additional forcing term on the right-hand side in (17). There will be a solution for the forced equation (17) that satisfies the boundary conditions (18) only if a certain compatibility condition is satisfied. We can obtain this compatibility condition for example by a direct construction of ζ_2.

Let us first define the linear operator $\mathcal{L}$ such that

$$\mathcal{L}(\phi) = \{\rho_0(c-u_0)^2\phi_z\}_z + \rho_0 N^2\phi. \tag{21}$$

Any pair of functions ψ and ϕ satisfying the lower boundary condition of the problem ($\psi(-h) = 0$ and $\phi(-h) = 0$) also satisfies

$$\int_{-h}^{z}\{\psi\mathcal{L}(\phi) - \phi\mathcal{L}(\psi)\}\mathrm{d}z = \rho_0(c-u_0)^2(\psi\phi_z - \phi\psi_z) = W(\psi,\phi;z) \tag{22}$$

where we have defined the Wronskian functional of ψ and ϕ as

$$W(\psi,\phi;z) \equiv \rho_0(c-u_0)^2(\psi\phi_z - \phi\psi_z). \tag{23}$$

If ψ and ϕ are solutions to the modal equation (12), then $\mathcal{L}\phi = \mathcal{L}\psi = 0$ so that

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \psi\mathcal{L}\phi - \phi\mathcal{L}\psi = 0 \tag{24}$$

Hence $W(\psi,\phi;z)$ is actually independent of z.

Now, for ζ_{2X} solution of the forced equation (17), we have

$$\phi\mathcal{L}(\zeta_{2X}) - \zeta_{2X}\mathcal{L}(\phi) = \phi M_2 = \frac{\mathrm{d}W(\phi,\zeta_{2X};z)}{\mathrm{d}z}, \text{ and} \tag{25}$$

and similarly for ψ. Integration of (25) and elimination of ξ_{2Xz} results in

$$\phi \int_{-h}^{z} \psi M_2 dz - \psi \int_{-h}^{z} \phi M_2 dz = \zeta_{2X} W(\psi, \phi; z), \tag{26}$$

The general solution of ζ_{2X} is then the sum of a general solution for the unforced problem plus the particular solution just identified,

$$\zeta_{2X} = A_{2X}\phi + \phi \int_{-h}^{z} \frac{M_2\psi}{W(\psi,\phi)} dz - \psi \int_{-h}^{z} \frac{M_2\phi}{W(\psi,\phi)} dz. \tag{27}$$

where we recall that $W(\psi, \phi)$ is constant. This general solution for ζ_{2X} was so far obtained by applying only the boundary condition on the bottom ($z = -h$). In the process, we have introduced another modal function ψ and the Wronskian, W. By applying the free surface boundary condition, and requiring that ψ be linearly independent of ϕ such that $\psi(0)$ does not satisfy the upper boundary condition of (17), we can now obtain a compatibility condition that is independent of ψ and W. Plugging the solution of ζ_{2X} into the free surface boundary condition in (18), we obtain

$$\rho(c-u_0)^2 \left\{ \phi_z \int_{-h}^{0} \frac{M_2\psi}{W} \mathrm{d}z - \psi_z \int_{-h}^{0} \frac{M_2\phi}{W} \mathrm{d}z \right\} - \rho_0 g \left\{ \phi \int_{-h}^{0} \frac{M_2\psi}{W} dz - \psi \int_{-h}^{0} \frac{M_2\phi}{W} \mathrm{d}z \right\} = N_2. \tag{28}$$

Recalling that $\rho(c-u_0)^2\phi_z = \rho_0 g\phi$ at $z = 0$ from (13), it then follows that the compatibility condition is

$$\int_{-h}^{0} M_2\phi \mathrm{d}z = [N_2\phi]_{z=0}. \tag{29}$$

Here we have obtained the compatibility condition through the direct construction of ζ_2. However, the compatibility condition can be obtained more easily without knowledge of ζ_2 (or higher-order ζ_n), and without the need to introduce an additional function ψ.

The compatibility condition is that the inhomogeneous terms in (17,18) should be orthogonal to the solution of the adjoint of the modal equations (12,13). This construction is fairly straightforward. We first begin by combining equations (22) and (25) into

$$\phi M_2 = \{\rho_0(c-u_0)^2(\phi\zeta_{2Xz} - \zeta_{2X}\phi_z)\}_z. \tag{30}$$

Integrating (30) and applying the free-surface boundary condition for the modal function ϕ (13) first, then the free-surface boundary condition (18) results in the compatibility condition (29) found earlier. This last method can easily be applied at any order in the expansion.

Note that the amplitude A_2 is left undetermined at this stage; further expansion into higher orders will result in an evolution equation for A_2. In general, solutions for ζ_{n+1} will result in a compatibility condition and thus, an evolution equation for A_n.

5 Korteweg-deVries (KdV) equation

Substituting the expressions for M_2 and N_2 (19, 20) into the compatibility condition (29), we obtain the evolution equation for $A(X,T)$, namely the KdV equation

$$A_T + \mu A A_X + \lambda A_{XXX} = 0, \tag{31}$$

where the coefficients μ (nonlinearity) and λ (dispersion) depend on the modal function ϕ:

$$I\mu = 3\int_{-h}^{0} \rho_0(c-u_0)^2\phi_z^3 \mathrm{d}z, \tag{32}$$

$$I\lambda = \int_{-h}^{0} \rho_0(c-u_0)^2\phi^2 \mathrm{d}z, \tag{33}$$

$$\text{where } I = 2\int_{-h}^{0} \rho_0(c-u_0)\phi_z^2 \mathrm{d}z, \tag{34}$$

The KdV equation (31) is solved with the initial condition $A(X, T=0) = A_0(X)$ where $A_0(X)$ is determined from linear long wave theory, and is in essence the projection of the original initial conditions onto the appropriate linear long wave mode. Localized initial conditions lead to (at sufficiently large time) the generation of a finite number of solitary waves, or internal solitons.

For waves moving to the right, where $c > u_M = \max u_0(z)$, I and λ are always positive. For the surface mode, $\phi(z) > 0$ and $\phi(0) = 1$ (no extrema in the interior) we see that $\mu > 0$. In general, μ can take either sign, and may be zero in some special situations. Explicit evaluation of the coefficients μ and λ requires knowledge of the modal function, and hence they are usually evaluated numerically. The modal function is known in several instances, and we illustrate the process with two simple examples.

5.1 Example 1: Surface water waves with no surface tension

For water waves, we set $\rho=$ constant so that $N^2 = 0$. We also assume that there is no background shear. The modal solution to equation (12) satisfying the boundary conditions (13) is then

$$\phi = \frac{z+h}{h} \text{ for } -h < z < 0,\ c = (gh)^{1/2}. \tag{35}$$

Note that there are no other modes in this system. Plugging in the modal function into equations (32-34), the coefficients I, μ and λ are

$$I = \frac{2\rho_0 c}{h} \text{ and } \mu = \frac{3c}{2h} \text{ and } \lambda = \frac{ch^2}{6}. \tag{36}$$

Thus the KdV equation for water waves is, in the original variables,

$$\zeta_t + c\zeta_x + \frac{3c}{2h}\zeta\zeta_x + \frac{ch^2}{6}\zeta_{xxx} = 0. \tag{37}$$

Note that here $z_m = 0$ so we identified A with $\zeta(x, 0, t)$, the free surface displacement, to leading order. For zero surface tension, this is the equation derived by Korteweg and de Vries in 1895 (and first by Boussinesq in the 1870's).

5.2 Example 2: Interfacial waves

For a two-layer fluid, waves may occur at the interface. Let the density be constant with value ρ_1 in an upper layer of height h_1 and ρ_2 in the lower layer of height $h_2 = h - h_1$. We assume the fluid is stably stratified such that $\rho_2 > \rho_1$. The density in the fluid is $\rho_0(z) = \rho_1 H(z + h_1) + \rho_2 H(-z - h_1)$ and the buoyancy frequency is $\rho_0 N^2 = g(\rho_2 - \rho_1)\delta(z + h_1)$. Here $H(z)$ is the Heaviside function and $\delta(z)$ is the Dirac δ-function. For simplicity, we assume that $\rho_1 \approx \rho_2$, the usual situation in the ocean. As mentioned earlier, the upper boundary condition for $\phi(z)$ then is approximately $\phi(0) \approx 0$. The modal function is then

$$\phi = \frac{z + h}{h_2} \text{ for } -h < z < -h_1, \ \phi = -\frac{z}{h_1} \text{ for } -h_1 < z < 0. \tag{38}$$

The corresponding coefficients are

$$\mu = \frac{3c(h_1 - h_2)}{h_1 h_2}, \ \lambda = \frac{c h_1 h_2}{6}, \ c^2 = \frac{g(\rho_2 - \rho_1)}{\rho_2}\frac{h_1 h_2}{h_1 + h_2}. \tag{39}$$

When the interface is closer to the free surface than to the bottom ($h_1 < h_2$), the nonlinear coefficient μ for these interfacial waves is negative. For single-layer water waves μ always remains positive. In the case when $h_1 \approx h_2$, μ nearly vanishes and it is necessary to introduce higher- order nonlinearity in order to balance the dispersion.

6 Extended Korteweg-deVries equation

As seen in the example of interfacial waves, the quadratic nonlinearity may vanish, and in this instance, it is necessary to use an extended KdV equation which contains higher-order nonlinearities and additional terms.

6.1 Higher-order expansions

Proceeding to the next highest order in the asymptotic expansion yields a set of equation analogous to (17,18) for ζ_3, whose compatibility condition then determines an evolution equation for the second-order amplitude A_2. Using the transformation $A + \varepsilon A_2 \to A$, and then combining the KdV equation (31) with the evolution equation for A_2 leads to a higher-order KdV equation

$$A_T + \mu A A_X + \lambda A_{XXX} + \varepsilon(\lambda_1 A_{XXXXX} + \sigma A^2 A_X + \mu_1 A A_{XXX} + \mu_2 A_X A_{XX}) = 0. \tag{40}$$

Explicit expressions for the coefficients are known analogs of (32-34). This higher-order KdV equation is Hamiltonian provided $\mu_2 = 2\mu_1$.

It is important to note that equation (40) is not unique: the near-identity transformation $A \to A + \varepsilon(aA^2 + bA_{XX})$ reproduces the same equation (40) to the same order in ε provided the coefficients are also changed to

$$(\lambda_1, \sigma, \mu_1, \mu_2) \to (\lambda_1, \sigma - a\mu, \mu_1, \mu_2 - 6a\lambda + 2b\mu).$$

Note that (40) at O(1) needs to be used to transform terms of the kind $\varepsilon A A_T$ and εA_{XXT}.

Furthermore, when $\mu \neq 0$, $\lambda \neq 0$, the enhanced transformation

$$A \to A + \varepsilon \left(aA^2 + bA_{XX} + a' A_X \int^X AdX + b' X A_T \right) \tag{41}$$

reduces (40) to the KdV equation. From a mathematical point of view, this implies that KdV is a normal form of the system, or in other words, the lowest order and simplest form characterizing the dynamics given the long-wave, small amplitude approximation made. Physically, this implies that for small perturbations with $\varepsilon \ll 1$, no additional dynamics are introduced by the higher-order terms.

6.2 Extended KdV equation

A particularly important case arises when the nonlinear coefficient μ is close to zero. In this case, the near-identity transformation (41) cannot cancel out the the cubic nonlinear term in the higher-order KdV equation (40) at this order. This identifies this particular higher-order term as being the most important one in balancing dispersion if $\mu \to 0$. The KdV equation (31) is then replaced by the extended KdV (or Gardner) equation,

$$A_T + \mu A A_X + \varepsilon \sigma A^2 A_X + \lambda A_{XXX} = 0. \tag{42}$$

For $\mu \approx 0$, a rescaling is needed, and the optimal choice is to assume μ is $\mathrm{O}(\delta)$, and then replace A with A/δ. The amplitude parameter is δ instead of δ^2. The resulting equation in the canonical form is

$$A_T + 6AA_X + 6\beta A^2 A_X + A_{XXX} = 0. \tag{43}$$

Like the KdV equation, the Gardner equation is integrable and can be solved using the Inverse Scattering method. The coefficient β can be either positive or negative, and the structure of the solutions depends on which sign is appropriate.

6.3 Solitary wave solutions

The solitary wave solutions for the extended KdV equation are given by

$$A = \frac{a}{b + (1-b)\cosh^2 \gamma(x - Vt)}, \tag{44}$$

$$\text{where } V = a(2 + \beta a) = 4\gamma^2, \; b = \frac{-\beta a}{(2 + \beta a)}. \tag{45}$$

There are two cases to consider. If $\beta < 0$, then there is a single family of solutions such that $0 < b < 1$ and $a > 0$. As b increases from 0 to 1, the amplitude a increases from 0 to a maximum of $-1/\beta$, while the speed V also increases from 0 to a maximum of $-1/\beta$. In the limiting case when $b \to 1$, the solution (44) describes the so-called "thick" or "table-top" solitary wave, which has a flat crest of amplitude $a_m = -1/\beta$ (see Figure 1). If $\beta > 0$, then the family of solutions allows both waves of depression and of elevation. In particular, there is a region of depression where solutions are not permitted, as indicated by the blue curve in Figure 1. As the amplitude is reduced, the solution becomes a "breather", a solitary wave with periodically-varying amplitude.

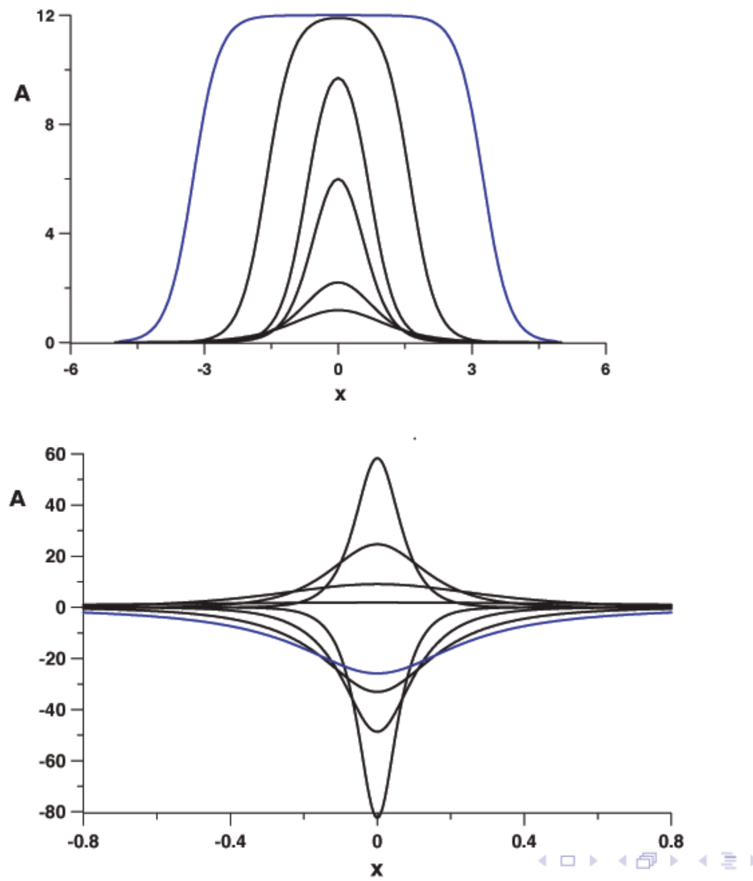

Figure 1: (top) Solitary wave solutions for $\beta < 0$. Note that there is a finite amplitude for the waves. Once this amplitude is achieved, the wave broadens and exhibits a “table top” behavior, indicated by the blue curve. (bottom) Solitary wave solutions for $\beta > 0$. Solutions can be both waves of elevation and depression. There is a minimum amplitude for waves of depression as indicated by the blue curve. Because of this, a breather solution is supported.

7 Other long-wave models

So far we have considered simple single-layer water wave examples. For a more realistic model of the ocean with a stratified near-surface layer lying above a deep ocean with constant density, a different scaling from the KdV equation is needed. In the surface layer, the long-wave scaling still holds, but this needs to be matched to a different scaling in the deep lower layer, where the vertical scale matches the horizontal scale. We therefore introduce a rescaled layer depth H, with $h = H/\delta$. In this scenario, the KdV equation is replaced by the intermediate long-wave (ILW) equation

$$A_\tau + \mu A A_X + \delta \mathcal{L}(A_X) = 0, \tag{46}$$

$$\text{where } \mathcal{L}(A) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k \coth kH \exp(ikX)\mathcal{F}(A)dk \tag{47}$$

$$\text{and } \mathcal{F}(A) = \int_{-\infty}^{\infty} A \exp(-ikX)dX. \tag{48}$$

Here the nonlinear coefficient μ is again given by (32) with $-h$ now replaced by $-\infty$, while the dispersive coefficient δ is defined by $I\delta = \{\rho_0 c^2 \phi^2\}_{z\to\infty}$.In the limit $H \to \infty$, $k \coth kH \to |k|$, the ILW equation (46) becomes the Benjamin-Ono (BO) equation. In the opposite limit, where $H \to 0$, then (46) reduces to a KdV equation. Both equations are integrable.

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