GFD 2013 Lecture 6: Stratified Environments

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1 Introduction

Buoyancy driven flows in stratified environments occur frequently in the real world. For example, when rivers flow into the ocean, the fresh river water floats and forms a gravity current within the stratified ocean. Another example the dust plume from the eruption of

Figure 1: The eruption of Eyjafjallajökull.

Eyjafjallajökull (Fig. 1), in which the hot, dust laden air rose until it reached the height of the air in the atmosphere with the same density and was then interleaved into the air column due to the stratification. This lecture explores the interleaving of intermediate density fluid into stratified fluid.

2 Intrusion in a two-layer fluid

As a simple example, we consider an intrusion in a two-layer fluid. Fig. 2 shows the conceptual setup. A layer of density ρ_i, where $\rho_U < \rho_i < \rho_L$, intrudes at the interface between the surface layer (density ρ_U) and the bottom layer (density ρ_L). Part of the intrusion flows in at a height above the two-layer interface and the other part penetrates below the interface.

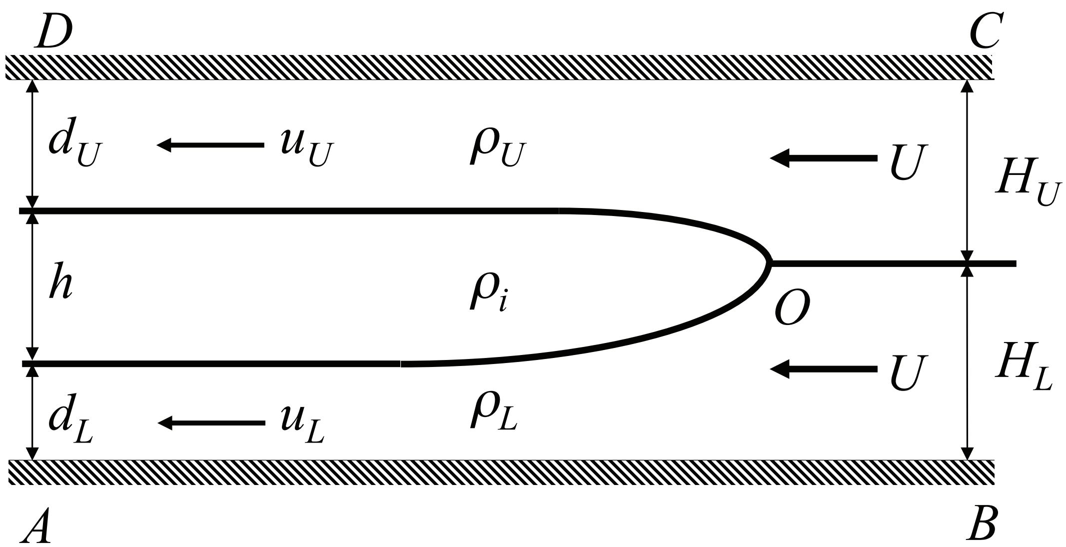

Figure 2: Schematic of an intrusion into a two-layer fluid. The density of the intrusion is ρ_i, where $\rho_U < \rho_i < \rho_L$, and in this frame of reference the intrusion is at rest.

In this example, it is useful to define three reduced gravities,

$$g'_{iU} = \frac{g\left(\rho_i - \rho_U\right)}{\rho_0} \tag{1a}$$

$$g'_{Li} = \frac{g\left(\rho_L - \rho_i\right)}{\rho_0} \tag{1b}$$

$$g'_{LU} = \frac{g\left(\rho_L - \rho_U\right)}{\rho_0} = g'_{iU} + g'_{Li}. \tag{1c}$$

These are reduced gravities for the three possible combination of the densities and are used in the following analysis.

2.1 The doubly symmetric case

The most obvious special case is a symmetric intrusion about the two-layer interface, where the intrusion density is the average of the layer densities. In this so-called doubly symmetric case we have

$$H_L = H_U = \frac{H}{2} \tag{2a}$$

$$\rho_i = \frac{\rho_U + \rho_L}{2}. \tag{2b}$$

This case is analogous to two gravity currents: a bottom current and a top current. Dimensional analysis yields that the front speed of the intrusion,

$$U = F\sqrt{\frac{g'_{iU}H}{2}} = \frac{1}{2}F\sqrt{g'_{LU}H}. \tag{3}$$

2.2 The general case

The general case is more complicated, with dimensional analysis giving

$$U = F\sqrt{g'_{iU}H}\, f\left(\frac{g'_{iU}}{g'_{Li}}, \frac{H_U}{H_L}\right), \tag{4}$$

where f is an arbitrary function of the dimensionless numbers $\frac{g'_{iU}}{g'_{Li}}$ and $\frac{H_U}{H_L}$. If we approach the problem from the viewpoint of Benjamin analysis and conserve mass, momentum and energy as in lecture 3, we can reduce the problem to two equations in two unknowns. They are nonlinear, and have multiple solutions. The equations and their solutions are described by Holyer and Huppert(1980) [1].

2.3 Equilibrium intrusions

We define equilibrium intrusions so that the fronts of the upper and lower component of the intrusion travel at the same speed, as in Fig. 2. We can construct an approximation for the relative heights of the layers and the density of the intrusion required for this situation to occur. Mass conservation requires

$$u_U d_U = U H_U \tag{5a}$$

$$u_L d_L = U H_L. \tag{5b}$$

Combining these with the equations of Holyer and Huppert gives

$$g'_{iU}\left(H_U - d_U\right) = g'_{Li}\left(H_L - d_L\right), \tag{6}$$

and

$$\frac{(H_U - d_U)}{H_U} = \frac{(H_L - d_L)}{H_L}. \tag{7}$$

We can use this simple constraint on the relative heights in the upper and lower part of the equilibrium intrusion problem to calculate a dynamical constraint on the intrusion density ρ_i. By resubstituting Eq. (7) into Eq. (6) we can obtain an equation that is linear in ρ_i so that

$$\rho_i = \frac{\rho_U H_U + \rho_L H_L}{H_U + H_L}. \tag{8}$$

This means that for equal velocities in the upper and lower part of the intrusion layer to be dynamically consistent with the density layering, we need to have an intrusion density that is the density layer height-weighted arithmetic mean of the upper and lower densities. Given the densities ρ_U, ρ_L and ρ_i, we can define h_E to satisfy Eq. (8) when $H_L = h_L$,

$$\hat{h}_E = \frac{h_E}{H} = \frac{g'_{iU}}{g'_{Li}}. \tag{9}$$

This definition can be used to introduce the nondimensional speeds for intrusions at the surface, U_H, the bottom, U_O and for equilibrium intrusions, U_E. We have

$$U_H = F\sqrt{1-\hat{h}_E} \tag{10a}$$

$$U_O = F\sqrt{\hat{h}_E} \tag{10b}$$

$$U_E = F\sqrt{\hat{h}_E(1-\hat{h}_E)} \tag{10c}$$

2.4 Minimization of Available Potential Energy

We can derive the speed at which the intrusion penetrates the layered stratification from simple energy conservation. In this case, we envision a lock release experiment of a fluid of density ρ_i into a two-layer stratification. After the system has equilibrated, and under the assumption that the fluids have stayed unmixed, we end up with a three-layer configuration. We consider the setup shown in Fig. 3a, in which the available potential energy, E_a is

$$E_a = \frac{1}{2}g\alpha\left(1-\alpha\right)\left(\left(H^2 - 2H_L H\right)\rho_i - \left(H - H_L\right)^2\rho_U\right). \tag{11}$$

We minimize the available potential energy with respect to H_L. This gives

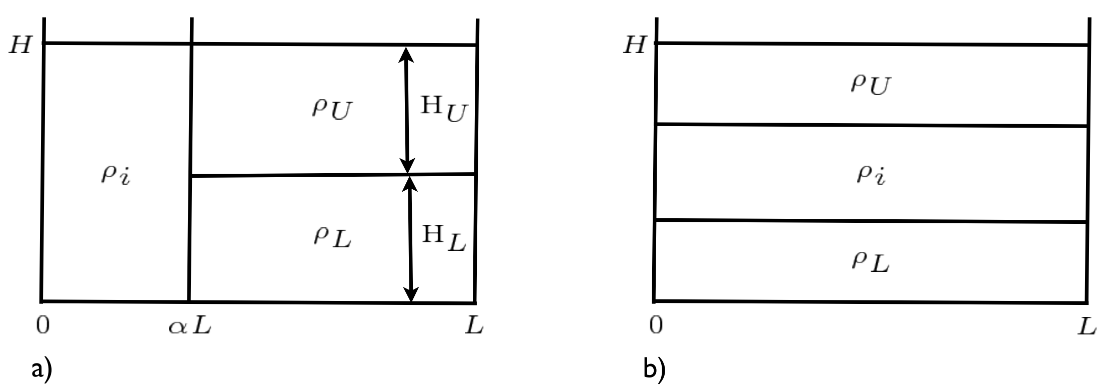

Figure 3: a) The initial setup and b) the final state.

$$\rho_i = \frac{\rho_U H_U + \rho_L H_L}{H}, \tag{12}$$

i.e. the equilibrium intrusion solution is identical to the case of minimum available potential energy. We turn again to the nondimensional velocity, which we define as an expansion in $\left(\hat{H}_L - \hat{h_E}\right)$ to second order,

$$\hat{U}^2 = \hat{U_E}^2\left(a\left(\hat{H}_L - \hat{h_E}\right)^2 + b\left(\hat{H}_L - \hat{h_E}\right) + c\right). \tag{13}$$

When $\hat{H_L} = \hat{h_E}$, $\hat{U} = \hat{U_E}$, so $c = 1$. We can solve for a and b using Eq. (10a) when $\hat{H_L} = 1$ and Eq. (10b) when $\hat{H_L} = 0$. Expanding Eq. (13) gives

$$\hat{U} = F\sqrt{\hat{H_L}^2 - 2\hat{H_L}\hat{h_E} + \hat{h_E}}, \tag{14}$$

so when $H_L = \frac{1}{2}$, $\hat{U} = \frac{1}{2}F$, i.e., when $H_L = H_U$, the speed of the intrusion is the always same, no matter what its density is. An explanation for this can be found by looking at a

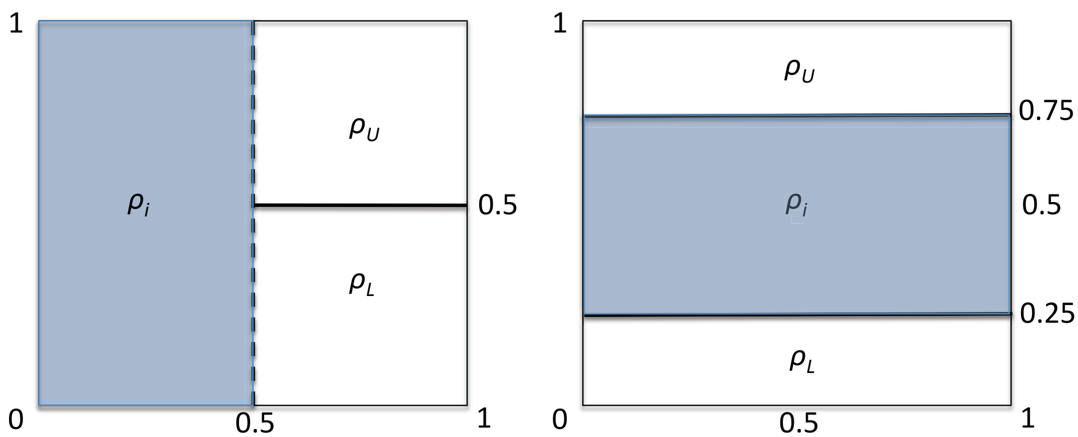

Figure 4: A special case of the setup shown above.

specific case of the setup, shown in Fig. 4 energy of the initial state is

$$\frac{1}{2}g\int_0^H \rho_i z \mathrm{d}z = \frac{1}{4}g\rho i H^2, \tag{15}$$

and the potential energy of the final state is

$$g\int_{\frac{H}{4}}^{\frac{3H}{4}} \rho_i z \mathrm{d}z = \frac{1}{4}g\rho i H^2, \tag{16}$$

i.e. the potential energy of the fluid in the intrusion does not change. This is equivalent to the height of the center of mass being conserved.

2.5 Multiple layer lock exchange

Let us consider the case of multiple layer lock exchange shown in Fig. 5. As before, we expect that

$$U \sim \sqrt{g'H}, \tag{17a}$$

$$\sim \sqrt{\delta\rho}, \tag{17b}$$

because this is just a multi-layer case of the interleaving flow discussed above. However, surprisingly, experiments yield

$$U \sim \frac{\delta\rho}{\Delta\rho}, \tag{18}$$

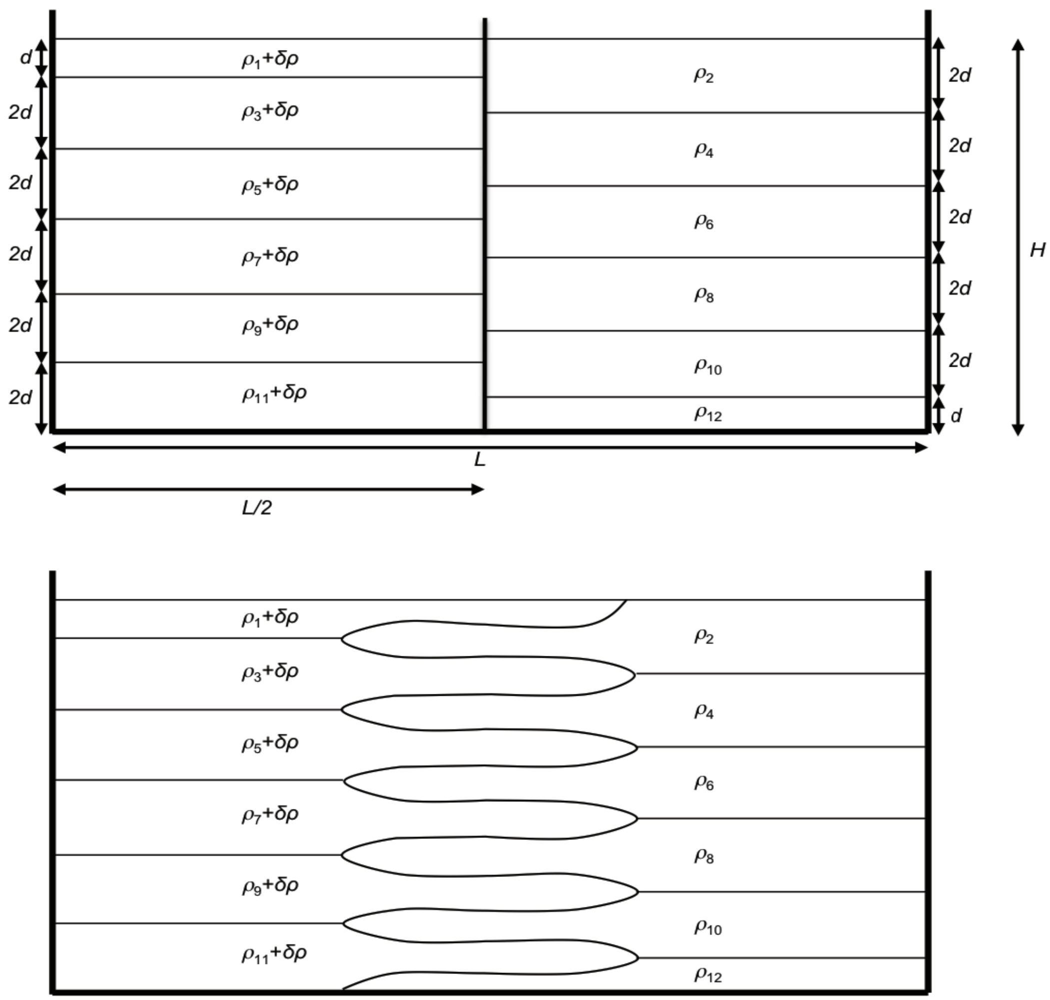

Figure 5: Schematic of intrusions in a stratified fluid. In this case, $\rho_i = \rho_{i-1} + \Delta\rho$ where $\Delta\rho$ is a constant. A small density difference is $\delta\rho < \frac{\Delta\rho}{2}$ is added to the densities on the left hand side. This provides an asymmetry, but all of the layers still interleave.

with no square root sign. If we calculate the available potential energy when $\delta\rho = 0$, we find that

$$E_a = \frac{1}{8} g \left(\rho_1 - \rho_N\right) d^2 \tag{19a}$$

$$= \frac{1}{8} g \left(N - 1\right) \Delta\rho d^2, \tag{19b}$$

where $2d$ is the depth of each layer and N is the number of layers. The available potential energy is equal to the kinetic energy of the whole system, so for each of the $N-1$ intrusions,

$$U_i \sim \sqrt{g \Delta\rho d}. \tag{20}$$

When $\delta\rho \neq 0$, there is extra potential energy from the asymmetry of the flow. The new available potential energy is

$$E_a = \frac{1}{8} g \left(\rho_1 - \rho_N\right) d^2 + \frac{1}{8} g \delta\rho d^2. \tag{21}$$

In this case, the available potential energy is again equal to the kinetic energy produced by the system, which can be divided into a shear component, where the shear velocity is U_s and a component from the velocity of the intrusions, U_i,

$$E_a \sim \left(\left(N - 1\right) U_i + U_s\right)^2. \tag{22}$$

If we assume that $U_s \ll U_i$, substitute in for U_i and expand, we find

$$U_s \sim \sqrt{gH} \sqrt{\frac{\delta\rho}{\rho_0}} \sqrt{\frac{\delta\rho}{\Delta\rho}} \sqrt{\frac{d}{H}}. \tag{23}$$

This explains Eq.(18), since $U_s \sim \delta\rho$.

References

[1] J. Y. Holyer and H. E. Huppert, *Gravity currents entering a two-layer fluid*, J. Fluid Mech., 100 (1980), pp. 739–767.