GFD 2013 Lecture 7: Continuously Stratified Ambient

Paul Linden; notes by Daniel Lecoanet and Karin van der Wiel

June 25, 2013

1 Introduction

In this lecture, we discuss the propagation of a gravity current into a continuously stratified medium, i.e., a background with a density $\rho(z)$, which is a continuous function of z. An important property of such a background is the buoyancy frequency (Brunt-Väisälä frequency)

$$N = \sqrt{-\frac{g}{\rho_0}\frac{d\rho}{dz}}, \tag{1}$$

which is the typical frequency of oscillation of a vertically displaced fluid element. Imagine a fluid element with volume V at an initial height z_i and density $\rho_i = \rho(z_i)$. If this fluid element is displaced to a position $z_i + s$, its density can be approximated by

$$\rho(z_i + s) \approx \rho(z_i) + \frac{d\rho}{dz}s. \tag{2}$$

Thus, Newton's second law gives

$$\rho(z_i)V\ddot{s} = -g\frac{d\rho}{dz}s, \tag{3}$$

which implies the fluid element oscillates at the buoyancy frequency (Equation 1). We will make the Boussinesq approximation, which implies that N is constant. This is only valid if $d\rho/dzL \ll \rho_0$, where L is the vertical length scale of the system, and ρ_0 is a typical density.

In these notes, we will mostly discuss the release of a fluid of constant density ρ_c into an ambient, stratified fluid with density $\rho(z)$. The behavior is very different depending on if there is a height h_N satisfying $\rho_c = \rho(h_N)$, or if instead ρ_c is either larger or smaller than $\rho(z)$ for all z in the domain. In the first case, there will be an intrusion at height $z = h_N$, where the fluid is locally neutrally buoyant. On the other hand, if ρ_c is larger (smaller) than $\rho(z)$, then then a gravity current forms on the bottom (top) of the domain. We will treat these cases separated. At the end of these notes, we will briefly mention the case of the release of a stratified fluid into a stratified ambient.

2 Gravity Current

To simplify our analysis, we will assume that ρ_c, the density of the released fluid, is larger than $\rho(z=0)=\rho_B$, the density at the bottom of the ambient fluid (the case of $\rho_c < \rho_0$, the density at the top of the ambient fluid is likely similar). In this case, a gravity current forms at the bottom of the domain.

The driving force for the gravity current is a horizontal pressure jump between the released fluid and the ambient. In hydrostatic balance, the pressure is given by the integral of the density, i.e., an average density. Thus, to lowest order, we would expect the gravity current to be equivalent to one released into a constant density ambient with density ρ_E, where this density is given by the average of ρ over the height of the current, i.e., $\rho_E = \rho(h/2)$ for Boussinesq stratification (see Figure 1). For an energy-conserving gravity current from a lock of depth D, we have $h = D/2$, so

$$\rho_E = \rho(D/4) = \rho_B - \frac{\rho_0}{4g} N^2 D. \tag{4}$$

We have

$$g'_E = g'_N + \frac{1}{4} N^2 D, \tag{5}$$

where

$$g'_N = g \frac{\rho_c - \rho_B}{\rho_0} \tag{6}$$

is the strength of the gravity current in terms of background stratification.

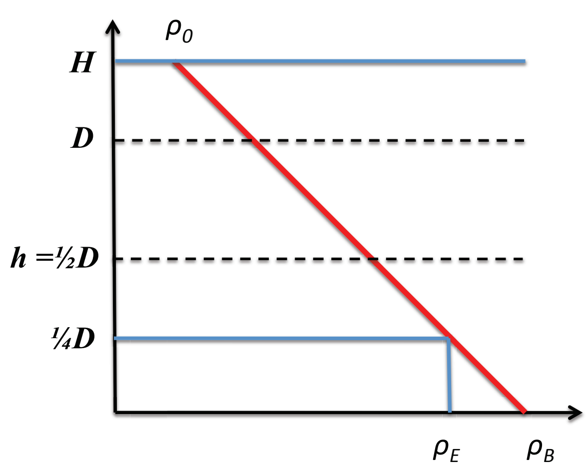

Figure 1: A schematic depicting the equivalent density ρ_E. This is the average density of the ambient stratified fluid over the bottom half of the domain. The simplest prediction for a gravity current with density $\rho_c > \rho_B$ released into a stratified fluid is that the gravity current acts as if it was released into a fluid with constant density ρ_E.

Recall that a gravity current released into a fluid of constant density has a Froude number

$$\frac{U}{\sqrt{g'D}} = \frac{1}{2}\sqrt{2 - D/H}. \tag{7}$$

If the gravity current is equivalent to one released into an ambient fluid with constant density, then this relation should still hold, with g' replaced by g'_E. This makes the prediction

$$F_N = \frac{1}{2}, \tag{8}$$

where

$$F_N \equiv \frac{U}{\sqrt{\left(g'_N D + \frac{1}{4}N^2 D\right)\left(2 - \frac{D}{H}\right)}}. \tag{9}$$

This prediction was tested experimentally and numerically for various values of ρ_c and stratifications. We introduce the stratification parameter

$$S \equiv \frac{\rho_B - \rho_0}{\rho_c - \rho_0}, \tag{10}$$

[5] recalling that ρ_0 is the density at the top of the domain. Thus, $S = 0$ for an unstratified ambient fluid, $S = 1$ if $\rho_c = \rho_B$, and $S > 1$ if there is an intrusion. Figure 2 shows F_N as a function of S. We note that there appears to be a critical S_C above which the Froude number deviates from one half.

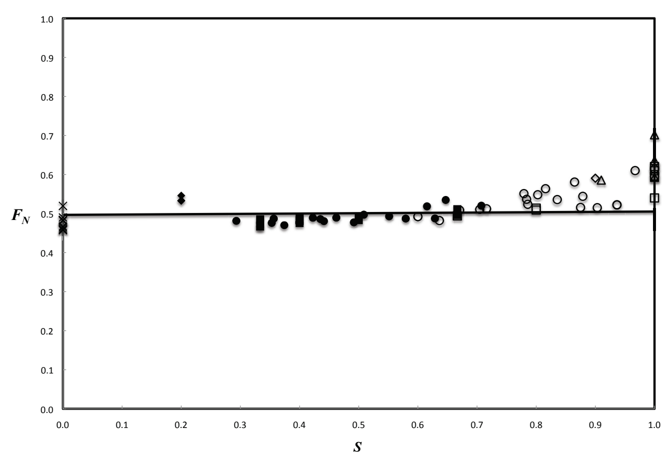

Figure 2: Froude number F_N as a function of the stratification parameter S for experiments and simulations from [2, 4, 3, 6]. Subcritical currents are depicted with open symbols and supercritical currents are depicted with filled symbols.

To understand the deviations from the $F_N = 1/2$, we must discuss internal waves. Internal waves are a generalization of the vertical oscillation of a fluid parcel in a stratified background. The momentum equation and buoyancy equation can be manipulated into the following equation for the vertical velocity w,

$$(\partial_x^2 + \partial_z^2)\partial_t^2 w + N^2 \partial_x^2 w = 0, \tag{11}$$

where we assume 2D flow. Assuming we can Fourier decompose w as

$$w = \hat{w} \exp(i(kx + mz - \omega t)), \tag{12}$$

we have the dispersion relation for internal waves

$$\frac{\omega^2}{N^2} = \frac{k^2}{k^2 + m^2}. \tag{13}$$

For long waves satisfying $m \gg k$, we have that the phase and group velocity are both given by

$$\frac{\omega}{k} = \pm \frac{N}{m}. \tag{14}$$

Now we will impose that the boundary conditions that $w = 0$ on the top ($z = H$) and bottom ($z = 0$). This implies

$$w \propto sin\left(\frac{n\pi z}{H}\right), \tag{15}$$

where n is an integer. This implies m is quantized

$$m = \frac{n\pi}{H}, \tag{16}$$

and the wave velocity is

$$c = \pm \frac{NH}{n\pi}. \tag{17}$$

The fastest wave speed is

$$c_{\max} = \frac{NH}{\pi}. \tag{18}$$

If the gravity current travels faster than this velocity, it is supercritical and cannot launch internal waves. However, for gravity current velocities smaller than $c_{\max}$, the current is subcritical and it radiates internal waves (see Figure 3). These internal waves change the upstream condition, and speed up the gravity current. The case $U = c_{\max}$ defines the critical stratification S_C. The curve S_C as a function of D/H is given in Figure 4. This is roughly consistent with the experimental results (Figure 2). Note that S_C can never be higher than ≈ 0.85.

There has been some work to apply hydraulic theory (i.e., an analysis similar to Benjamin) to the release of constant density fluid into a stratified ambient. However, this work has assumed that ρ becomes constant above the current, which is not the case. This approach requires a description of the flow which cannot be predicted by the theory.

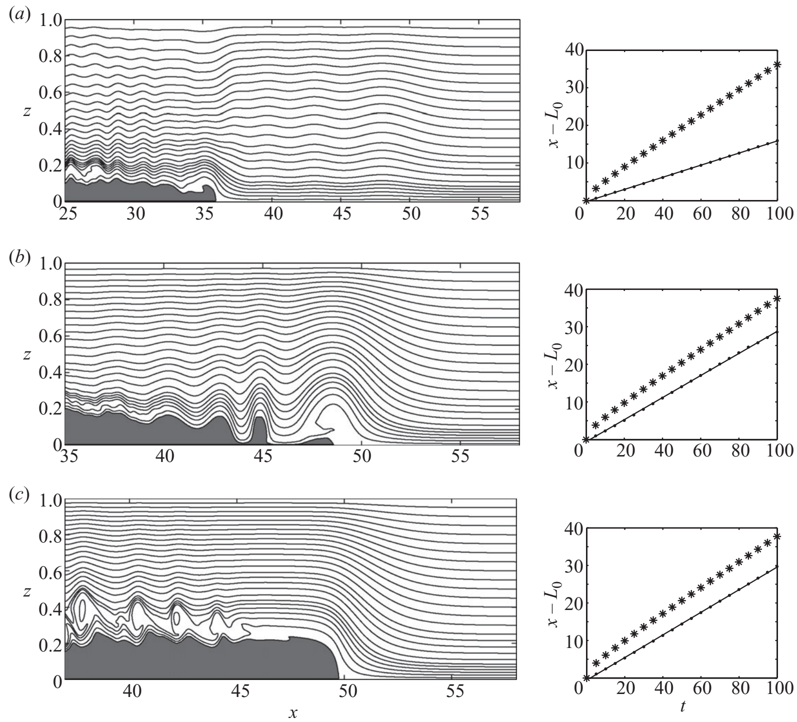

Figure 3: Numerical simulations of constant density gravity currents released into stratified ambient fluid. (a) & (b) show upstream propagating internal waves, whereas (c) does not [6].

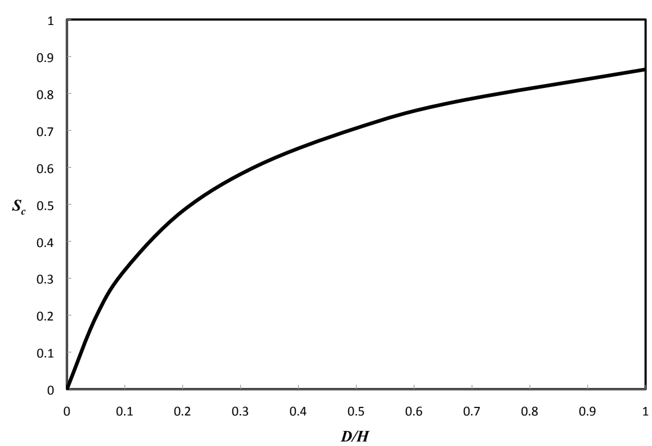

Figure 4: Curve of critical stratification parameter S_C as a function of D/H. For $S < S_C$, the gravity current is supercritical and does not emit upstream propagating internal waves, and the current evolves as if it is expanding into a medium of uniform density ρ_E. For $S > S_C$, the gravity current is subcritical and emits upstream propagating internal waves, influencing the upstream condition, and increase propagation speed of the current.

3 Intrusion Currents

As before we will consider the release of a finite volume of well-mixed fluid (ρ_i) into a linearly stratified channel (constant N^2). However, now we will consider the case where $\rho_u < \rho_i < \rho_l$. In this case a gravity intrusion current will form, propagating within the ambient fluid (see Figure 5). Assume the depth of the lock D is equal to the depth of the channel H.

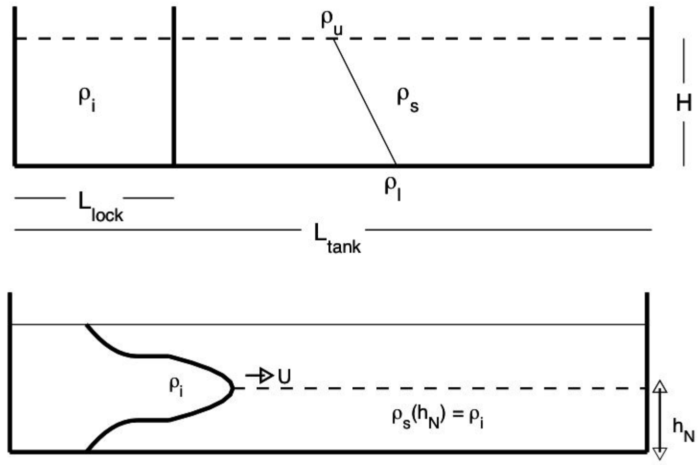

Figure 5: Top: A sketch of the initial setup for the intrusion where $\rho_u < \rho_i < \rho_l$. Bottom: A sketch of an intruding current with $\rho_i = 1/2(\rho_u + \rho_l)$ after the lock has been removed. In this case the neutral buoyancy level is at $h_N = H/2$.

The level of neutral buoyancy (h_N) is the level at which the density of the intrusion is equal to the density of the ambient fluid $\rho_i = \rho_s(h_N)$. This is the level along which the gravity current will propagate. Let us start with the special case where the density of the intrusion fluid is equal to the average density of the ambient fluid in the channel ($\rho_i = 1/2(\rho_u + \rho_l)$). The level of neutral buoyancy will be $h_N = H/2$. As before we use the Froude number to describe the flow and use the equivalent reduced gravity in place of g' (Equations 8 and 9). The strength of the gravity current in terms of the background stratification at $h = h_N$ is

$$g'_N = g\frac{(\rho_i - \rho_s(h_N))}{\rho_0} = 0. \tag{19}$$

With $H = D$ we find:

$$F_N = \frac{U}{1/2NH} = 1/2 \tag{20}$$

$$U = 1/4NH \tag{21}$$

The mid-depth current travels at half the speed of a gravity current along the bottom of a channel in a stratified fluid.

The total Available Potential Energy (APE) for the intrusion current is:

$$APE = g\int_{h_N}^{0} (\rho_s(z) - \rho_i) z dz + g\int_{0}^{H-h_N} (\rho_i - \rho_s(z)) z dz \tag{22}$$

$$= \frac{g}{3}\left((\rho_l - \rho_i)h_N^2 + (\rho_i - \rho_u)(H - h_n)^2\right). \tag{23}$$

The level h_N for which this has a minimum (differentiate with respect to h_N) is:

$$h_N = \frac{H}{2}. \tag{24}$$

In Figure 6 images of an intrusion gravity current are shown. The top panel shows the initial setup, with dye marking isopycnal surfaces. For this experiment N was chosen to be 1 s^{-1} and $h_N = 0.8$. In the next panels the evolution in time is shown, a gravity current develops. Note that in front of the current the isopycnal surfaces are deflected downwards, indicating waves are able to travel faster than the current and change the conditions of the ambient fluid. The flow is thus subcritical to at least one long wave mode.

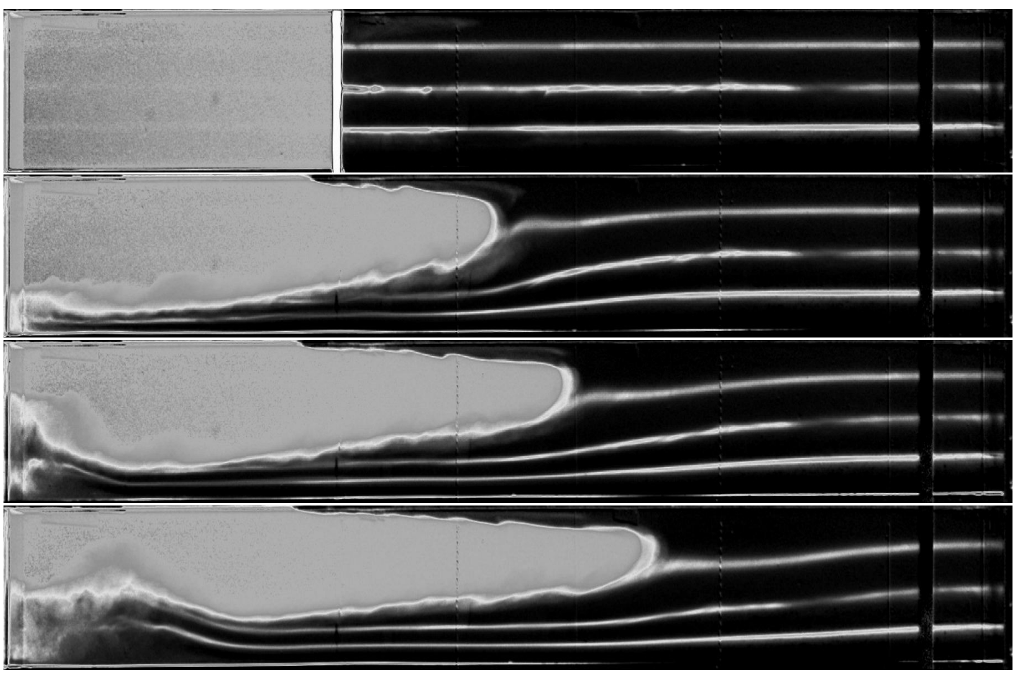

Figure 6: Images of an intrusion gravity current from experiments. In this case $N = 1$ and $h_N = 0.8$.

Figure 7 shows the dimensionless intrusion current velocity for different values of h_N. The minimum is the discussed special case of $h = 1/2H$, where $U = 1/2F_N NH$. At the boundaries where the current flows along the bottom and top ($h = 0$, $h = 1$) we find double the velocity ($U = F_N NH$). The dashed, horizontal grey lines in the figure show the speed of the first three modes of long waves. The grey line near the top of the plot is the first long wave mode. Its nondimensional speed is larger than the nondimensional current speed for all h and thus these flows are always subcritical to the first long wave mode. The second long wave mode is the middle grey line plotted. Depending on the value of h the flow is either subcritical (small and large h) or supercritical (mid-level currents) to this mode. The third long wave mode is too slow to make an impact for all h, so is unable to transmit information ahead of the intrusion current.

Based on these numbers it can be concluded that the experiment in Figure 6 is subcritical to the first mode of long waves only and thus it must be these waves that cause the deflections of the isopycnal surfaces in the ambient fluid ahead of the current.

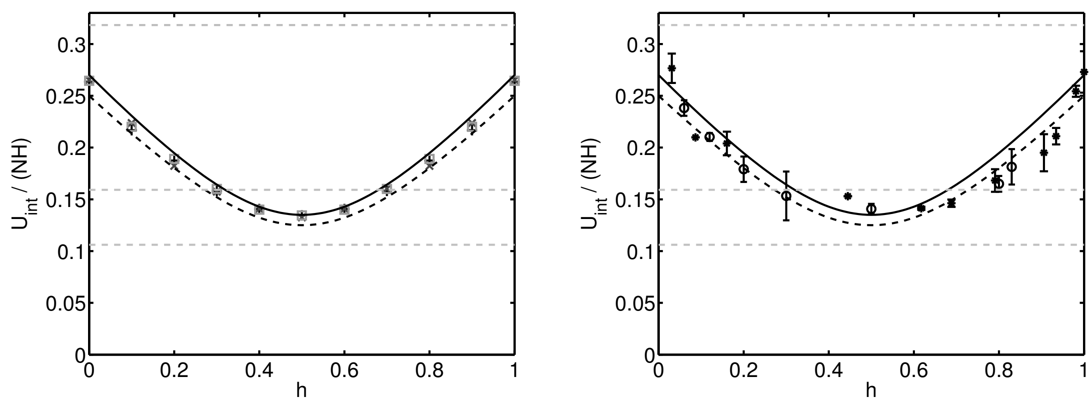

Figure 7: Comparison of dimensionless intrusion velocity ($U/(NH)$) for numerical simulations (left) and experiments (right) to model predictions. The dashed line shows the theoretical $F = 0.25$, the solid line $F = 0.266$. The light grey horizontal dashed lines on each plot represent from top to bottom the first, second and third mode long wave speed respectively.

4 Adjacent stratified regions

Finally, we consider the case of a lock exchange between two differently linear stratisfied fluids (Figure 8). The fluid in the left lock has a constant buoyancy frequency equal to N_i, the right lock has a constant buoyancy frequency N_a. Let us define the stratification ratio S as the ratio between the buoyancy frequency in the two locks:

$$S = \frac{N_i^2}{N_a^2}. \tag{25}$$

From previous lectures and the considerations above we already know the intrusion current speed for some cases.

- Equal stratification in the two locks, i.e. $N_i = N_a$ and thus $S = 1$. In this case there is no horizontal density gradient in the channel and thus no gravity current: $U = 0$.
- No stratification in the left lock, i.e. $N_i = 0$, $S = 0$. This is the case described in section 3, of a well-mixed fluid, propagating as an intrusion into a linearly stratisfied fluid, $U = 1/8NH$.

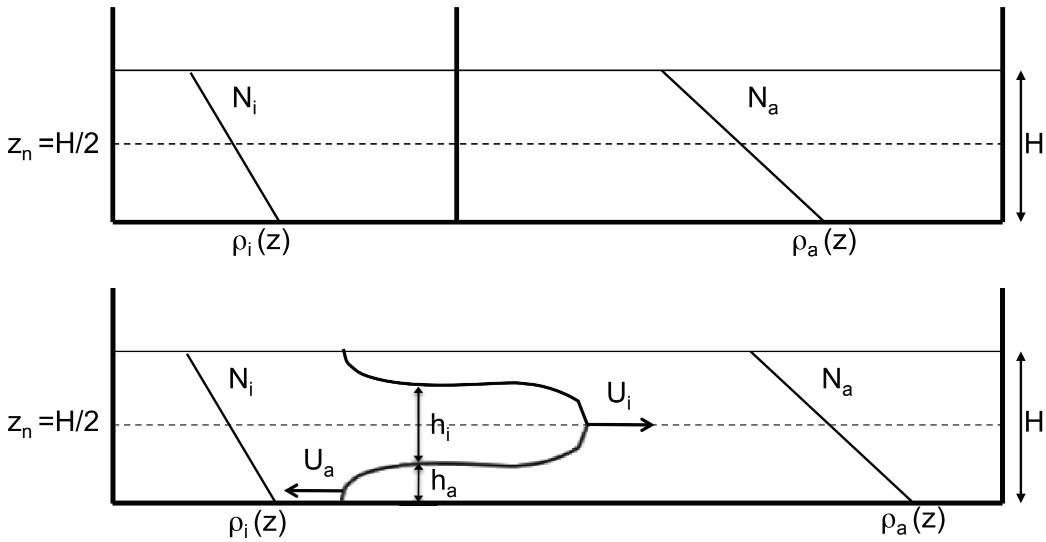

Figure 8: Schematic showing the initial conditions of lock-release of a linearly stratified intruding fluid of N_i, into a linearly stratified ambient fluid of N_a, where the average densities of both fluids are equal. From [1].

The cases where $0 \leq S \leq 1$, the intrusion current speed is a function of the stratification ratio: $U = 1/4NHf(S)$. Because N_i is smaller than N_a the intrusion current will travel from the left lock into the right fluid.

Figures 9 and 10 show experiments and numerical simulations respectively for two different stratification cases. On the left $S \simeq 0.2$, a relatively fast intrusion current, and on the right a slower current of $S \simeq 0.8$.

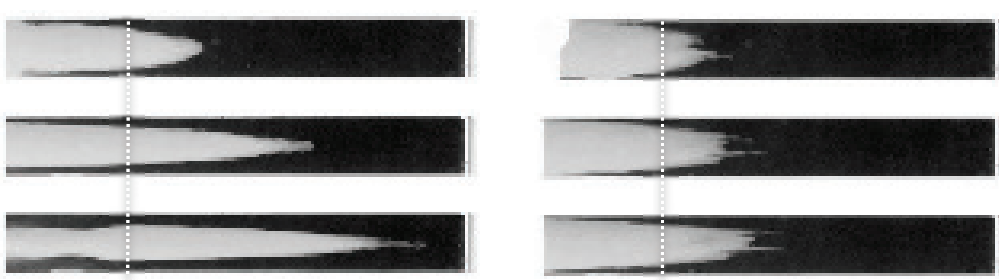

Figure 9: Snapshots of the laboratory experiments for $S = 0.23$ (left) and $S = 0.77$ (right) for $N_a = 1.5$ s^{-1} at the dimensionless times $Nat = 10$, 20 and 30. The dashed white line denotes the initial position of the gate. The intrusion fluid is visualized with dye.

References

[1] B. D. Maurer, D. T. Bolster, and P. F. Linden, *Intrusive gravity currents between two stably stratified fluids*, J. Fluid Mech., 647 (2010), pp. 53–69.

Figure 10: Snapshots of the numerical simulations for $S = 0.20$ (left) and $S = 0.80$ (right) for $N_a = 1\ \mathrm{s}^{-1}$ at the same dimensionless times as Figure 9. The dashed gray line denotes the initial position of the gate. The motion of the lock fluid is visualized using a passive tracer.

[2] T. Maxworthy, J. Leilich, J. E. Simpson, and E. H. Meiburg, *The propagation of a gravity current into a linearly stratified fluid*, J. Fluid Mech., 453 (2002), pp. 371–394.

[3] E. H. Meiburg, V. K. Birman, and M. Ungarish, *On gravity currents in stratified ambients*, Phys. Fluids., 19 (2007).

[4] J. O. Shin, S. B. Dalziel, and P. F. Linden, *Gravity currents produced by lock exchange*, J. Fluid Mech., 521 (2004), pp. 1–34.

[5] M. Ungarish and H. E. Huppert, *On gravity currents propagating at the base of a stratified ambient*, J. Fluid Mech., 458 (2002), pp. 283–301.

[6] B. L. White and K. R. Helfrich, *Gravity currents and internal waves in a stratified fluid*, J. Fluid Mech., 616 (2008), pp. 327–356.