Lecture 9 - Nonlinear waves in a variable medium

Lecturer: Roger Grimshaw. Write-up: Hélène Scolan

June 17, 2009

1 Introduction

The usual Korteweg-De-Vries equation, which assumes a uniform background state, is not sufficient to describe internal solitary waves in the coastal ocean. Indeed, the topography can vary horizontally, and the waves produced are not clean wave trains. This can be seen for example in the measurements of currents in the Australian Northwest Shelf reproduced in Figure 1.

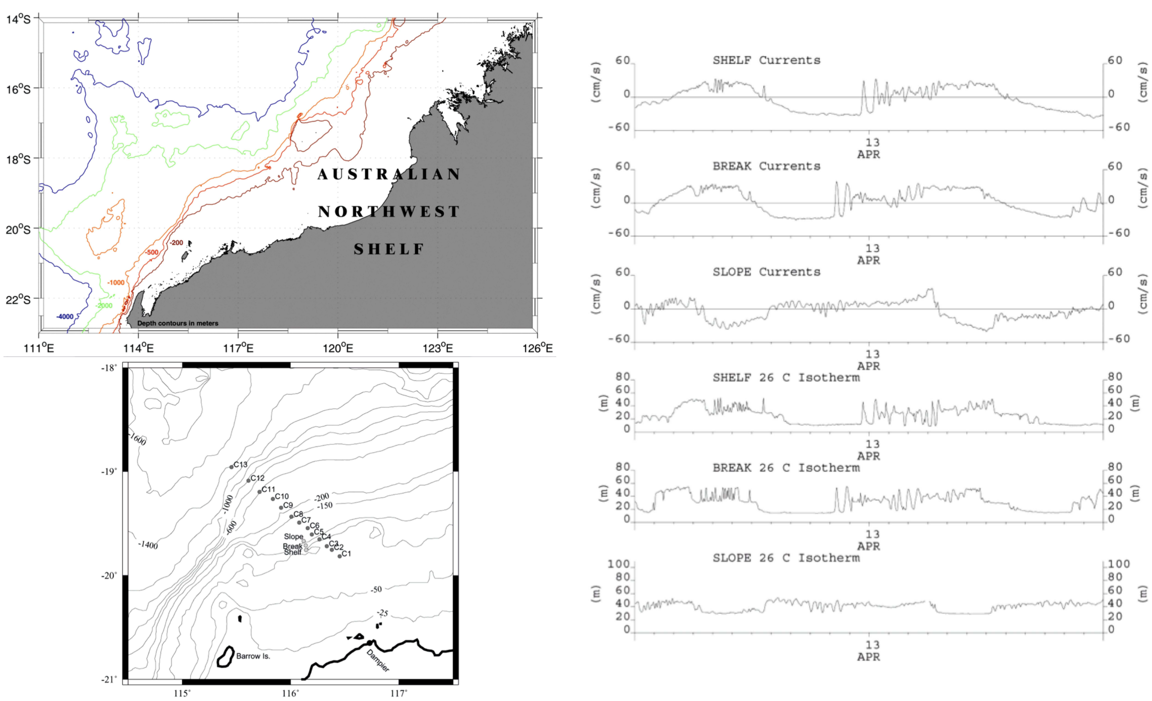

Figure 1: Time series of isotherm displacements and onshore currents are shown from 3 moorings, (Slope, Break and Shelf), located in 78 to 109 m water depths, and a few kilometers apart at the outer edge of the Australian Northwest continental shelf. The plots show a variety of nonlinear wave forms including bores on both the leading and trailing faces of the long internal tide, as well as short period (approximately 10 minutes, close to the buoyancy period) internal solitary waves. [After Holloway and Pelinovsky, 2001]. *An Atlas of Oceanic Internal Solitary Waves (February 2004) by Global Ocean Associates Prepared for Office of Naval Research Code 322 PO*

By incorporating a variable medium in the model, one can build the variable coefficient Korteweg-de Vries equation and find asymptotic and numerical solutions of the problem.

2 Waves in inhomogeneous medium

2.1 Linear waves and wave-action conservation law

First, we recall the properties of linear waves propagating through an inhomogeneous medium. Because of the presence of the variable background, the usual wave equation is modified. In many examples, the wave equation becomes

$$u_{tt} - (c^2(x)u_x)_x = 0 \tag{1}$$

where the wave speed $c(x)$ varies with position (for example $c(x) = \sqrt{gh(x)}$ for waves supported by the shallow-water equations, see Lecture 8).

We assume that the medium is "slowly varying" which means that the lengthscale L over which the medium changes is greater than the typical wavelength λ ($\lambda \ll L$) and so the coefficient $c(x)$ can be considered almost constant on the wave scale ($c = c(\epsilon x)$ with $\epsilon \ll 1$). The WKB approximation consists in looking for a solution close to the solution for a homogeneous medium $a \exp(-i\omega(t - x/c))$. Using an ansatz of the form $a(x,t)\exp(-i\omega(t - \tau(x)))$ with $\tau(x) = \int \frac{\mathrm{d}x}{c(x)}$, and developing an asymptotic expansion in the powers of ϵ, it can be shown that $a(x,t) \propto \frac{1}{\sqrt{c(x)}}$ (proof in [5] for example).

More generally (i.e. for wave equations not necessarily in the form of (1)), the WKB asymptotic solution can be written: $u \approx a(x,t)f(t - \tau(x)))$ where $\tau(x) = \int \frac{\mathrm{d}x}{c(x)}$ and where the phase $t - \tau(x)$ is assumed to vary rapidly compared with the amplitude function $a(x,t)$ and the speed $c(x)$. Then it can be shown that $a(x,t)$ verifies:

$$(a^2)_t + (ca^2)_x = 0 \tag{2}$$

This equation is called the wave action conservation law for wavetrains in slowly spatially varying medium.

The most general form of the wave action conservation law, for waves propagating in a non-uniform, time-dependent medium which may also sustain a mean flow $\mathbf{U}$ is

$$\left(\frac{E}{\hat{\omega}}\right)_t + \left(\frac{E}{\hat{\omega}}c_g\right)_x = 0, \tag{3}$$

where E is the wave energy density (which is usually related, but no necessarily equal to the square of the wave amplitude), $\hat{\omega}$ is the intrinsic frequency defined by

$$\hat{\omega} = \omega - \mathbf{k} \cdot \mathbf{U},$$

(i.e. the frequency of the wave seen by an observer moving with a mean flow if there is one) and c_g the group velocity. For more information about this conservation law see [2]. The quantity $E/\hat{\omega}$ is called the wave action density, so that the wave action flux is $\frac{E}{\hat{\omega}}c_g$ ($c_g \approx c$ if the medium is weakly dispersive).

This final equation can be interpreted in the following way: in the limit where $\hat{\omega}$ is constant (i.e the medium does not vary with time or there is no mean flow) then the wave action conservation law reduces to an energy conservation law. If $\hat{\omega}$ is not constant, then energy density is not conserved, but the wave action density $E/\hat{\omega}$ is.

2.2 Non linear waves: variable-coefficient KDV equation

To describe nonlinear internal waves in a variable medium, we begin with the basic non linear KdV equation:

$$u_t + cu_x + \mu u u_x + \beta u_{xxx} = 0 \tag{4}$$

in which we introduce the possibility of a variable background. Thus the linear phase speed and the coefficients μ and β have a spatial dependency. Furthermore, another term is needed if we want the variable-coefficient KdV equation to verify the general wave-action conservation law (3) in the limit where non-linear terms are negligible:

$$u_t + c(x)u_x + \frac{cQ_x}{2Q}u + \mu(x)uu_x + \beta(x)u_{xxx} = 0. \tag{5}$$

This additional term is written so that in the linear wave theory $u_t + c(x)u_x + \frac{cQ_x}{2Q}u = 0$ is transformed into $(\frac{Qu^2}{c})_t + (Qu^2)_x = 0$. Then, Qu^2 can be interpreted as the wave action flux (and so the wave-action density is $\frac{Qu^2}{c}$ since $c \approx c_g$ in the long wave/weak dispersion limit). The exact expression for Q depends on the original physical problem considered.

To maintain the balance between terms in the new equation including the effect of variable medium (5), we need the dispersion term, the non-linearity and the weak inhomogeneity term to be of the same order of magnitude. If $\frac{\partial}{\partial x} \sim \epsilon \ll 1$ and we suppose $u \sim \epsilon^a$ and $\frac{Q_x}{Q} \sim \epsilon^b$, the terms will be of the same order of magnitude if $a+b = 2a+1 = a+3$ which gives $a = 2$ and $b = 3$. So $\frac{Q_x}{Q}$ scales as ϵ^3. This implies that the variable-medium KdV is only valid in the limit where the medium varies very slowly compared with the horizontal scale of the wave.

As in the homogeneous KdV, it is useful to recast the governing equations in a moving coordinate system which follows the propagation of the wave, i.e. perform a change of variable in which $\xi \sim x/c - t$ where c is the phase speed. Here, the procedure is slightly more complex since c may vary with position. By analogy with the WKB approximation technique, we introduce

$$\tau = \int_0^x \frac{dx'}{c(x')},$$

This change of coordinate can be viewed as a mapping of the original spatial coordinate into a time-like coordinate, since the new variable τ is simply the travel time between the original position of the wave and its present location. If we also use the change of variable $X = \tau - t$, we then have

$$u_t = -u_X \tag{6}$$

$$u_x = \frac{u_X}{c} + \frac{u_\tau}{c}. \tag{7}$$

When the background varies very slowly compared with the size of the wave, it can be demonstrated that $\partial/\partial\tau \ll \partial/\partial X$, so that

$$u_x \simeq \frac{u_X}{c}. \tag{8}$$

Then, within the balance seen before the equation can be written:

$$u_\tau + \frac{Q_\tau}{2Q}u + \frac{\mu}{c}uu_X + \frac{\beta}{c^3}u_{XXX} = 0. \tag{9}$$

The two equations (5) and (9) are asymptotically equivalent and differ just by terms of $O(\epsilon^7)$. It is interesting and important to note that the coefficients μ, β and c now vary with the time-like variable τ. Physically this simply models the fact that as the wave propagates through the inhomogeneous medium, it “sees” a slowly time-dependent, but nearly homogeneous background around itself.

The more commonly used form of the KdV equation is obtained by putting $A = Q^{1/2}u$ which gives the variable-coefficient KdV equation:

$$A_\tau + \alpha A A_X + \lambda A_{XXX} = 0 \qquad \text{(vKDV)}$$

where $\alpha = \dfrac{\mu}{c\sqrt{Q}}$ and $\lambda = \dfrac{\beta}{c^3}$.

It can be verified that the variable-coefficient KdV has two conservation laws:

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} A\mathrm{d}X = 0, \tag{10}$$

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} A^2\mathrm{d}X = 0. \tag{11}$$

They are often referred as conservation of “mass” and momentum even if these are not the physical ones. The latter equation is in fact the conservation of the wave action flux. The former is asymptotically that for the physical mass.

3 Slowly varying periodic waves

3.1 Asymptotic expansion

As we suppose that the medium is slowly varying, we write $\alpha = \alpha(T)$ and $\lambda = \lambda(T)$ with $T = \sigma\tau$ with $\sigma \ll 1$. We can develop a multiscale expansion in powers of the small parameter σ for a modulated periodic wave by looking for solutions of the kind:

$$A = A_0(\theta, T) + \sigma A_1(\theta, T) + ...$$

where A is periodic in the phase $\theta = k(X - \frac{1}{\sigma}\int^T V(T)dT)$ with a fixed period of 2π, where k is a fixed constant and V remains to be determined.

As $\frac{\partial A_i}{\partial \tau} = \frac{\partial A_i}{\partial \theta}\frac{\partial \theta}{\partial \tau} + \frac{\partial A_i}{\partial T}\frac{\partial T}{\partial \tau} = A_{i\theta} \cdot (-\frac{k}{\sigma}V(T)) + A_{iT} \cdot \sigma$ and $\frac{\partial A_i}{\partial X} = kA_{i\theta}$, the expansion introduced in the vKDV gives:

$$O(\sigma^0): \qquad -VA_{0\theta} + \alpha A_0 A_{0\theta} + \lambda k^2 A_{0\theta\theta\theta} = 0 \tag{12}$$

$$O(\sigma): \qquad -VA_{1\theta} + \alpha(A_0A_1)_\theta + \lambda k^2 A_{1\theta\theta\theta} = -\frac{1}{k}A_{0T} \tag{13}$$

These are ordinary differential equation in θ with T as a parameter. A solution of (12) can be written:

$$A_0 = a\{b(m) + \text{cn}^2(\gamma\theta; m)\} + d,$$

where

$$b = \frac{1-m}{m} - \frac{E(m)}{mK(m)}, \quad \alpha a = 12m\lambda\gamma^2 k^2, \quad V = \alpha d + \frac{\alpha a}{3}\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\}.$$

This solution is a typical *cnoidal wave* $\text{cn}(x; m)$, which is a Jacobian elliptic function of modulus m ($0 < m < 1$). The functions $K(m)$ and $E(m)$ are the elliptic integrals of the first and second kind, a is the amplitude and d is the mean value of A over one period $\gamma = K(m)/\pi$, and the spatial period is $2\pi/k$.

This solution contains three free parameters which depend on T: for example the amplitude a, the mean level d and the modulus m. We can consider the two limit cases:

- $m \to 1$: This is the solitary wave case.

  Indeed, $b \to 0$ and $\text{cn}^2(x) \to \text{sech}^2(x)$. $\gamma \to \infty$ and $k \to 0$ with $\gamma k = K$ held fixed.

- $m \to 0$: This gives sinusoidal waves of small amplitude $a \sim m$ and wavenumber k.

3.2 Modulation equations

To completely describe the solution we must now find how a, d and m depend on the slow variable T. There are two methods for this: the Whitham avering method or the asymptotic expansion continued at higher level.

3.2.1 Whitham averaging method

The method is developed as follows:

- Step 1: Determine the three conservation laws for the vKDV equation.
- Step 2: Insert the periodic cnoidal wave into the conservation laws.
- Step 3: Average over the phase θ.

Conservation laws: We already have the mass and momentum conservation laws (10) and (11):

$$\frac{\partial}{\partial T}\int_0^{2\pi} A\mathrm{d}\theta = 0 \quad \text{and} \quad \frac{\partial}{\partial T}\int_0^{2\pi} A^2\mathrm{d}\theta = 0. \tag{14}$$

Since we are dealing with slowly varying waves, an additional conservation law is derived from the law of "conservation of waves" (or "conservation of crests"). Indeed for slowly varying waves we have the definitions $k = \frac{\partial\theta}{\partial X}$ and $\omega = -\frac{\partial\theta}{\partial t}$ so that:

$$k_T + \omega_X = 0.$$

Since ω does not depend on X, k is constant.

Substitution of cnoidal wave into conservation laws.

- The mass equation implies that the mean level d is constant.
- After averaging over θ, the momentum equation produces a relationship between a and m:

$$a^2\left\{\frac{1}{2\pi}\int_0^{2\pi}\mathrm{cn}^4(\gamma\theta;m)\mathrm{d}\theta - b(m)^2\right\} = \text{constant}$$

$$\Rightarrow \quad \frac{a^2}{m^2}\left\{(2-3m)(1-m) + \frac{(4m-2)E(m)}{K(m)} - 3m^2 b(m)^2\right\} = \text{constant}$$

  which uniquely determines the evolution of the modulus m:

$$F(m) \equiv K(m)^2\{(4-2m)E(m)K(m) - 3E(m)^2 - (1-m)K(m)^2\} = \text{constant}\frac{\alpha^2}{\lambda^2}$$

  since α and λ vary with T in a known way.

  The function $F(m)$, as seen in Figure 2 for example, is usually a monotically increasing function of m so if α/λ increases, m increases too. This implies that if the dispersive coefficient λ tends to zero then m tends to 1 and the waves become more like solitary waves.

  For example for water waves, $c = \sqrt{gh}$, $Q = c$, $\mu = 3c/2h$ and $\beta = ch^2/6$ which leads to $\alpha/\lambda \propto h^{-9/4}$ and $F(m) \propto h^{-9/2}$. As the wave approaches the beach $h \to 0$ and $m \to 1$, which means that the wave gradually transforms into a solitary wave. It's amplitude goes as $h^{-3/4}$ so that the surface elevation varies as h^{-1}.

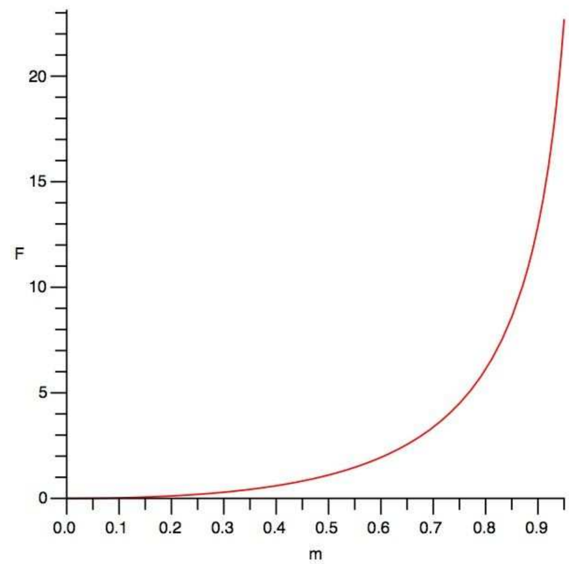

Figure 2: $F(m)$ in the case of water waves.

3.2.2 Asymptotic expansion continued

To find the conservation laws, we can also alternatively continue the method of asymptotic expansion to the next order. As A_1 must be periodic, we must force the right-hand side of (13) to be orthogonal to the periodic solutions of the adjoint to the homogeneous operator on the left-hand side (see the discussion for this point in Lecture 6).

Indeed let us define the operator L as

$$L = -V\frac{\partial}{\partial\theta} + \alpha(A_0.\ )_\theta + \lambda k^2 \frac{\partial}{\partial\theta^3}.$$

Then (13) can be written: $L(A_1) = F$ where $F = -\frac{A_{0T}}{k}$.

By definition, the adjoint L^* verifies for any periodic function B and A_1:

$$< BL(A_1) >=< A_1 L^*(B) >$$

where $< \quad >= \int_0^{2\pi} d\theta$. Thus if B is a solution of $L^*(B) = 0$, $< BF >=< BL(A_1) >=< A_1 L^*(B) >= 0$.

The adjoint equation $L^*(B) = 0$ is :

$$-VB_\theta + \alpha A_0 B_\theta + \lambda k^2 B_{\theta\theta\theta} = 0$$

$B = 1$ and $B = A_0$ are two periodic solutions of this equation. A third solution can be found but is not periodic. So we have two conditions $< 1(-\frac{A_{0T}}{k}) >= 0$ and $< A_0(-\frac{A_{0T}}{k}) >= 0$ which coincide to the statement that d is constant and to the momentum conservation law equation (14).

4 Slowly varying solitary waves

It is important to note that the results for a slowly-varying periodic wave cannot directly be extrapolated to the solitary-wave case: indeed, the limit $m \to 1$ requires $k \to 0$ and so the period becomes infinite. Accordingly, the condition that the local period $(1/kV)$ should be much smaller than the scale of the variable medium $(1/\sigma)$ is no longer satisfied.

Thus, we must refine the definition of "slowly-varying" for the case of solitary waves. The solitary wave will be considered slowly-varying if the half-width is much less than the scale of the variable medium $(1/\sigma)$. An asymptotic expansion can then be developed in the same way as before but with a new expression for the phase:

$$\phi = X - \frac{1}{\sigma}\int^{T} V(T)dT.$$

A is not required to be periodic in ϕ, and is defined in $-\infty < \phi < \infty$ and bounded in $\phi \to \pm\infty$. Without changing the problem we can choose $\lambda > 0$ so that small-amplitude waves propagate in the negative x-direction (a transposition A, x with $-A$, $-x$ gives the other side). We can also assume $A \to 0$ as $\phi \to \infty$ without imposing anything in the other boundary condition as $\phi \to -\infty$.

The resulting ODEs of the asymptotic expansion are:

$$-VA_{0\phi} + \alpha A_0 A_{0\phi} + \lambda A_{0\phi\phi\phi} = 0 \tag{15}$$

$$-VA_{1\phi} + \alpha (A_0 A_1)_\phi + \lambda A_{1\phi\phi\phi} = -A_{0T}. \tag{16}$$

The solution A_0 is now a solitary wave: $A = a\text{sech}^2(K\phi)$ with $V = \frac{\alpha a}{3} = 4\lambda K^2$ and only has one free parameter (a for instance). A background d can be added, but is constant and can be removed by a Galilean transformation.

At the next order, we require that $A_1 \to 0$ as $\phi \to \infty$. This imposes a new compatibility equation $< B(-A_{0T}) >= 0$ where $<>= \int_{-\infty}^{\infty} \mathrm{d}\phi$ and with the adjoint equation $L^*(B) = 0$:

$$-VB_\phi + \alpha A_0 B_\phi + \lambda B_{\phi\phi\phi} = 0.$$

Among the two possible bounded solutions $B = 1$ and $B = A_0$, only the latter satisfies the condition $A_1 \to 0$ as $\phi \to \infty$. So there is only one orthogonality condition which can be imposed which corresponds to the right-hand side of (16) being orthogonal to A_0 ie $< A_0.(-A_{0T}) >= 0$ ie :

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty} A_0^2 \mathrm{d}\phi = 0. \tag{17}$$

Even though there is only one equation this time, it is enough to determine the evolution of the free-parameter a. Substituting the sech^2 solution into the condition (17) yields

$$a^3 = \text{constant}\frac{\alpha}{\lambda}$$

which agrees with the limit $m \to 1$ of the periodic wave case.

4.1 Trailing shelf

A problem nevertheless occurs with the preceeding derivation since the vKDV equation has two conservation laws (momentum and mass) whereas only one condition can be imposed (17), which happens to coincide with the momentum equation (11). This means that for solitary waves we can not simultaneously require conservation of the total mass and momentum. This can also be seen by examining the solution of (16) for A_1: indeed if we integrate in ϕ, with the boundary condition $A_1 \to 0$ as $\phi \to \infty$ and $A_1 \to H_1$ as $\phi \to -\infty$ and using the properties of A_0 we get:

$$VH_1 = -\frac{\partial M_0}{\partial T} \quad \text{where} \quad M_0 = \int_{-\infty}^{\infty} A_0 \, \mathrm{d}\phi \quad \text{and} \quad H_1 = \frac{6}{\alpha K}\frac{a_T}{a} \tag{18}$$

which illustrates how the "total mass" changes as the solitary wave propagates.

The solution to this problem consists in constructing a "trailing shelf" A_s such that $A = A_0 + A_s$. A_s is of small amplitude $O(\sigma)$ but with a long length-scale $O(1/\sigma)$ which has $O(1)$ mass but $O(\sigma)$ for the momentum. It is located behind the solitary wave and to leading order has a value independent of T so that $A_s = \epsilon A_s(X)$ with $X = \sigma x$ for $X < \phi(T) = \int^{T} V(T)dT$.

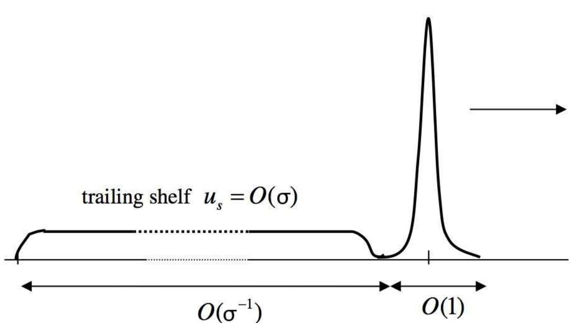

Figure 3: Trailing shelf residing behind the solitary wave.

The trailing shelf is determined by its value at the location $X = \phi(T)$ of the solitary wave, in particular $A_s(\phi(T)) = H_1(T)$. It can have a negative or positive polarity depending on the sign of λa_T and so on the growth or decay of the wave amplitude. It may be verified that the slowly-varying solitary wave and the trailing shelf together satisfy conservation of mass. Continuing the expansion to higher orders in σ reveals how the shelf itself evolves and generates secondary solitary waves.

4.2 Critical case

If we reconsider the expression for the free parameter a:

$$a^3 = \text{constant}\frac{\alpha}{\lambda}$$

we see that there is a critical point when $\alpha = 0$ where we may expect a dramatic change in the wave structure. Indeed, the wave amplitude goes to 0 if $\alpha \to 0$, and decreases as $|\alpha|^{1/3}$

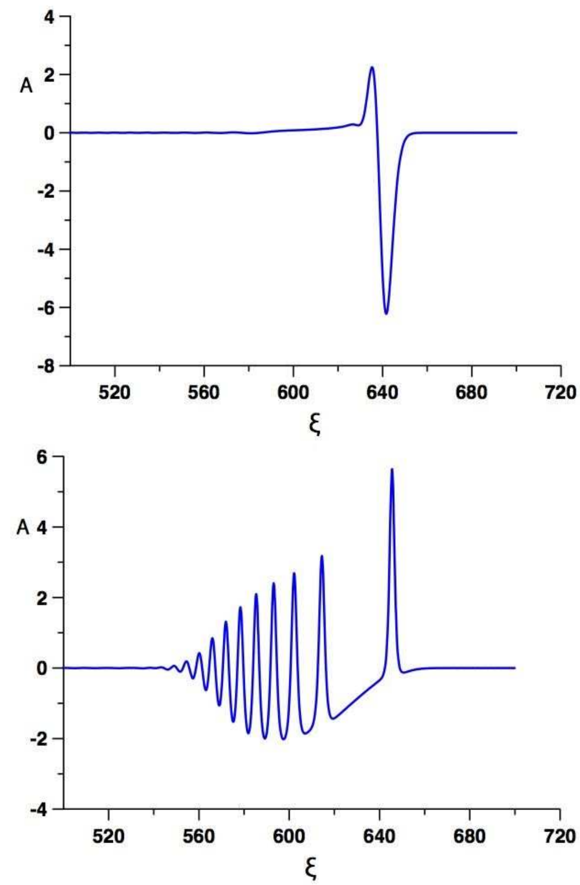

Figure 4: $\lambda = 1$ and α varies from -1 to 1. Upper panel: $\alpha = 0$. Lower panel: $\alpha = 1$

while the mass M_0 of the solitary wave only grows as $|\alpha|^{-1/3}$. Meanwhile, the amplitude A_s of the trailing shelf grows as $|\alpha|^{-8/3}$ with the opposite polarity of the wave.

Essentially the trailing shelf passes through the critical point as a disturbance of the opposite polarity to that of the original solitary wave, which then being in an environment with the opposite sign of α, can generate a train of solitary waves of the opposite polarity, riding on a pedestal of the same polarity as the original wave. Figure 4 shows for instance the possibility of conversion of a depression wave (with a positive shelf) into a train of elevation waves riding on a negative pedestal. The mean level of the new wave-train is negative corresponding to the initial negative mass of the depression wave.

References

[1] *Magic lectures on nonlinear waves.* http://www.maths.dept.shef.ac.uk/magic/course.php?id=21. (Mathematics Access Grid Instruction and Collaboration).

[2] F. Bretherton and C. Garrett, *Wavetrains in inhomogeneous moving media.*, Proc. Roy. Soc., 302 (1968), pp. 529–54.

[3] R. Grimshaw, *Internal solitary waves in a variable medium.*, Gesellschaft fur Angewandte Mathematik, 30, pp. 96–109.

[4] R. Grimshaw, E. Pelinovsky, and TalipovaT., *Modeling internal solitary waves in the coastal ocean..*, Surveys in Geophysics, 28 (2007), pp. 273–298.

[5] J. Simmonds and M. J.E., *A first look at perturbation theory*, Dover Publications, Inc. Mineola, New York, 1997.