Models of Volcanic Tremor and Singing Icebergs

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Abstract

We explore the possibility of flow instability through an elastic channel to explain seismic observations of tremors in such varied environments as volcanoes and icebergs. We consider the flow to be constricted in a narrow conduit and subjected to perturbations due to the propagation of compressional waves in the larger conduit. The pressure wave in the fluid are generated at the outlet of the channel by the oscillation of the walls. The growth of these self-excited perturbations may be a source of nonlinear oscillations of the wall in the channel. We explore the linear stability of various models driven by pressure gradients in the channel. We find that no linear instability persists without inertial effects in the channel. For finite yet small Reynolds number, we find growing oscillatory modes corresponding to pressure variations in the fluid flowing through the channel. Without inertial effects, the decaying oscillations in the channel have frequencies close to the natural notes of the conduit, corresponding to sound waves bouncing back and forth in the conduit. With inertial effects, however, the apparent frequencies of growing oscillatory modes are smaller than the corresponding harmonic frequencies of the conduit. The smaller apparent frequency of growing modes compared to the normal modes of a magma chamber (or reservoir, for water systems) relaxes the constraints on the linear dimension of the inferred finite bodies that host the tremor source. The proposed self-excited model may be a relevant candidate to explain observations of tremors in various geologic contexts, including at deep and shallow depths in volcanic plumbing systems and in complex water circulation in icebergs. Inertial effects in fluid flow may be responsible for the observed frequency drift which is a characteristic of tremors.

1 Introduction

Volcanoes and, as recently observed icebergs [15], are generating long period seismic activity that can persist for several minutes up to a few months. The so-called harmonic tremors are characterized by their low frequency content with a peaked spectra between 0.15 and 10 Hz. Within volcanoes, where they are generally observed, tremors often occur in connection with eruptions [2, 3] and their source originates from the shallow depth to about 50 km.

Volcanic tremors have attracted considerable attention because of their potential to constrain physical processes occurring inside volcanoes plumbing system. Tremors have been observed in numerous volcanoes including Mount Kilauea in Hawaii [16, 2], Mt. Etna [9] and Mt. St. Helens [10]. For a recent review of the observational evidence, we refer to [14]. The driving mechanism behind tremor activity is believed to involve a complex interaction between magmatic flow and surrounding bedrock in the volcano pipe system [1, 11, 3] as opposed to the brittle failure of rock that characterizes tectonic earthquakes. Tremors appear to be primarily composed of P waves [2].

Early theoretical models of tremors considered the vibration of compressible fluid-filled cracks in a layered elastic crust [7]. Aki et al. [1, 2] modeled the source of tremors to be the vibration of a crack filled with magma driven by the excess magmatic pressure. Such a jerky extension of the crack sets up a vibration, with a predominant period proportional to the linear dimension of the crack, and the amplitude proportional to the excess pressure and to the area of extension. Chouet [8] explored this assumption further with a model consisting of three elements, namely a triggering mechanism -an explosive point-source overpressure-, a resonator, and a radiator. The low frequency of tremors can be explained by normal mode oscillation in a volcanic magma chamber. For instance, the extremely long-period volcanic tremor, with periods up to 7s, observed at Mount Aso may be generated by a fluid-filled crack of modest size, a magma body 0.5 m thick and 0.5 km long [11]. The large linear dimension of required magma body to explain low frequency signals might be found in certain volcanic contexts, but may not be found systematically at shallow depths. Julian [13] showed that certain aspects of volcanic tremors such as periodic and chaotic oscillations, correlation between tremor activity and surface eruption, changes in amplitude, or frequency drifting can be explained by the flow of an incompressible viscous fluid through a channel with movable elastic walls, noting further that the obtained nonlinear process finds analogies with the excitation mechanism of musical wind instruments [4, 5]. Balmforth et al. [6] explored the stability of an incompressible flow through elastic conduits buried in a Hookean solid. They found that a critical Reynolds number is required for growing instability, and that Instabilities analogous to roll waves occur in this system.

In this study, we look for a mechanism for self-excitation of oscillations. We start with the model of Balmforth et al. [6] consisting of fluid flow through an elastically deformable channel but we couple the flow to sound waves in a larger conduit. The compressional waves in the conduit modulate the pressure at the outlet of the channel. We explore the possibility of such system to generate growing instabilities. The advantage of the proposed mechanism compared to previous models is that no outside perturbation, such as jerky over-pressure, a nearby earthquake or sudden crack opening is required to start the oscillations. In a first section, we develop the equations relevant to incompressible fluid flow in a narrow conduit with deformable walls. We obtain equations similar to lubrication theory but with inertia terms conserved. We consider the linear stability of simple models when inertial effects are ignored. In a next section, we consider the linear stability of dynamic model and look for a critical Reynolds number for onset of growing oscillations. In order to simplify an analytic treatment of the stability analysis, we consider the case of an open-funnel geometry. In an appendix, we derive the governing equations for the case of a compressible fluid.

2 Model and Governing Equations

We consider the two-dimensional flow of a Newtonian fluid in conduit of width W being constricted into a channel of height H and length L (see Fig. 1) such as $H \ll L$ and $H \ll W$. In the channel, where $0 < x < 1$, the fluid velocity $\mathbf{v} = (u\,\hat{\mathbf{e}}_x + v\,\hat{\mathbf{e}}_y)$ satisfies the Navier-Strokes equation

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \tag{1}$$

subjected to the equation of mass conservation for incompressible material

$$\nabla\cdot\mathbf{v} = 0 \tag{2}$$

and where p is the fluid pressure, ρ is the fluid density and ν is the kinematic viscosity. We consider the boundary conditions

$$\begin{aligned} u(x,h,t) &= 0 \\ v(x,0,t) &= u_y(x,0,t) = 0 \\ h(0,t) &= H \end{aligned} \tag{3}$$

where $h(x,t)$ is the unknown height of the channel walls. The second condition in eq. (3) is due to the symmetry about the x-axis. In the larger conduit, $L < x < L + l$, we assume that the flow is steady and only perturbed by a pressure wave, satisfying

$$p_{tt} - c^2\nabla^2 p = 0 \tag{4}$$

where c is the sound speed in the fluid. The higher fluid velocity in the channel generates pressure fluctuations at the exit of the constricted conduit. The pressure variations are then propagated as elastic waves in the larger conduit. The wave bounces back to the channel and perturbs the mean flow, generating potential instabilities. Let us define the dimensionless parameters

$$\begin{aligned} \tilde{x} &= L\,x, & \tilde{y} &= H\,y, \\ \tilde{u} &= U\,u, & \tilde{v} &= \frac{U\,H}{L}\,v, \\ \tilde{p} &= \rho\,\nu\frac{U\,L}{H^2}\,p, & \tilde{t} &= \frac{L}{U}\,t \\ \tilde{c} &= C & \tilde{l} &= L\,l \\ \tilde{\rho} &= \rho_0\,\rho & \tilde{W} &= H\,W \end{aligned} \tag{5}$$

where the (soon-abandoned) tilde-decorated variables have physical dimensions. Using the Reynolds number

$$R = \frac{H^2 U}{L\,\nu} \tag{6}$$

and the dimensionless relative velocity (sound wave speed compared to flow speed) for the conduit

$$\alpha = \frac{C}{U} \tag{7}$$

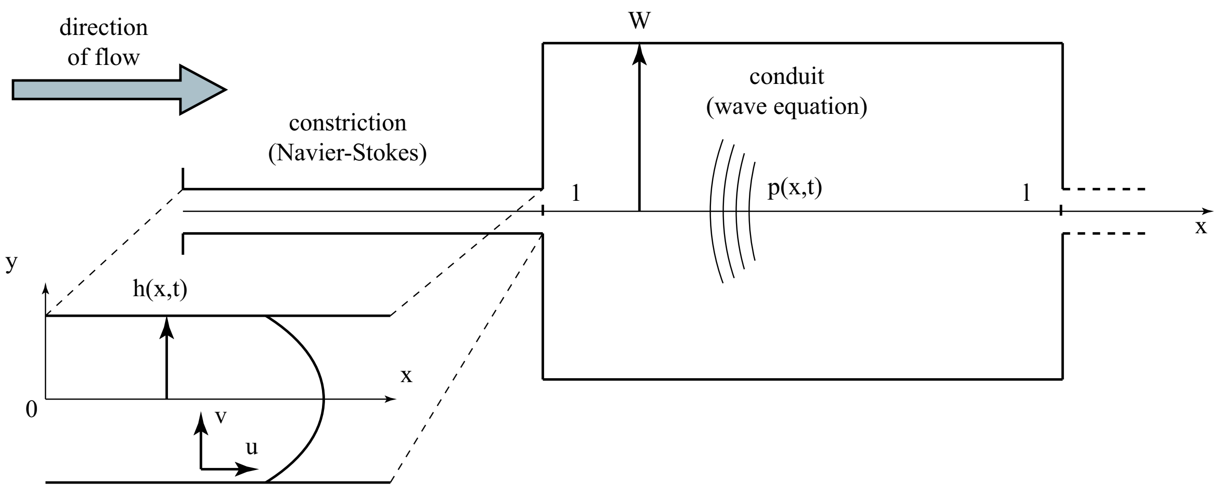

Figure 1: Geometry of the constricted flow. Fast fluid in the constricted channel generates a pressure wave in the conduit. The pressure wave bounces back to the channel and perturbs the mean flow, generating potential instability.

The dimensionless governing equations in the conduit are the wave equation

$$p_{tt} - \alpha^2 p_{xx} = 0 \tag{8}$$

and the equation of mass conservation

$$R\, u_t + p_x = 0 \tag{9}$$

Assuming $H/L \ll 1$, the two-dimensional Navier-Stokes equation in the channel simplifies to

$$\begin{aligned} R\,(u_t + u\,u_x + v\,u_y) &= -p_x + u_{yy} \\ p_y &= 0 \end{aligned} \tag{10}$$

so one writes $p = p(x,t)$. The boundary conditions are now

$$\begin{aligned} u(x,h,t) = v(x,0,t) = u_y(x,0,t) = 0 \\ h(0,t) = 1 \end{aligned} \tag{11}$$

For steady-state condition, one simply have $u_{yy} - p_x = 0$. Integrating twice and using boundary conditions of eq. (11), one obtains

$$u(\mathbf{x},t) = \frac{1}{2} p_x (y^2 - h^2) \tag{12}$$

The average across-slot velocity

$$\bar{u}(x,t) = \frac{1}{h} \int_0^h u\, dy \tag{13}$$

is given by integration of eq. (12)

$$\begin{aligned}\bar{u}(x,t) &= \frac{p_x}{2h}\int_0^h (y^2-h^2)\,dy \\ &= \frac{p_x}{2h}\left[\frac{1}{3}y^3 - h^2 y\right]_0^h = -\frac{p_x}{3}h^2\end{aligned} \tag{14}$$

The mean horizontal velocity is directly related to the pressure gradient and the wall height.

Integrating the divergence-free condition on the velocity field, one finds

$$\begin{aligned}\int_0^h \nabla\cdot\mathbf{v} &= \int_0^h u_x + v_y\,dy = \int_0^h u_x\,dy + \int_0^h v_y\,dy \\ &= \int_0^h u_x\,dy + [v]_0^h\end{aligned} \tag{15}$$

Defining $v(h) = h_t$, one gets

$$h_t + \int_0^h u_x\,dy = \frac{\partial}{\partial x}\int_0^h u\,dy - u(h)\,h_x + h_t = 0 \tag{16}$$

but the no-shear boundary condition $u(h) = 0$ (eq. 11) leads to

$$h_t + \frac{\partial}{\partial x}\int_0^h u\,dy = 0 \tag{17}$$

or, using the mean velocity formulation of eq. (13), one gets the expression for the conservation of mass in terms of mean horizontal flow $\bar{u}$ and height h of the channel walls

$$h_t + (h\bar{u})_x = 0 \tag{18}$$

or in terms of pressure using eq. (14)

$$h_t = \frac{1}{3}\left(h^3 p_x\right)_x \tag{19}$$

Assuming that a linear relationship exists between pressure and the wall height in the channel, eq. (19) corresponds to a nonlinear diffusion equation, which also appears in shallow water approximations [12].

2.1 Simple Models

For the sake of simplicity, we assume that the elastic wall height is uniform at equilibrium (no flow assumption). We consider that the elastic walls exert a force on the fluid proportional to the change of height (i.e., a mattress of springs approximation)

$$p = \Gamma(h-1) \tag{20}$$

where $h = 1$ is considered the dimensionless height of the wall at equilibrium (uniform height approximation). Variable Γ is dimensionless is the corresponding physical quantity is

$$\tilde{\Gamma} = \rho\,\nu\frac{U\,L}{H^3}\,\Gamma \tag{21}$$

Using eqs. (20), (14) and (18), one obtains a nonlinear diffusion equation for the wall height

$$h_t = \frac{\Gamma}{3}\left(h^3 h_x\right)_x \tag{22}$$

We consider now the perturbation of the system by a sound wave propagating in the larger cavity from $x = 1$ to $x = l$. We adopt a one-dimensional approximation for the wave propagation. Using the dimensionless equation of state

$$p = \alpha^2 R\,\rho\ , \tag{23}$$

the conservation of mass

$$\rho_t + u_x = 0 \tag{24}$$

and the conservation of momentum

$$R\,u_t + p_x = 0\ , \tag{25}$$

one obtains the pressure wave equation

$$p_{tt} - \alpha^2 p_{xx} = 0 \tag{26}$$

subjected to the boundary condition $u(x = l, t) = 0$ or, equivalently, to $p_x(x = l, t) = 0$.

2.1.1 Dispersion Equation

We write the perturbation as

$$\begin{aligned} h &= 1 + h' e^{\lambda t + mx}, & 0 \le x \le 1 \\ p &= e^{\lambda t} P(x), & 1 \le x \le l + 1 \end{aligned} \tag{27}$$

where m and λ are complex and $h' \ll 1$. We further require that the pressure is continuous at the exit of the channel. Plugging eq. (27) into the nonlinear diffusion equation of eq. (22), and one obtains after linearization

$$\lambda = \frac{1}{3}\Gamma\, m^2 \tag{28}$$

The wall height can be written

$$h = 1 + h_1'\, e^{+x\sqrt{3\lambda/\Gamma} + \lambda t} + h_2'\, e^{-x\sqrt{3\lambda/\Gamma} + \lambda t} \tag{29}$$

but boundary condition at the origin $x = 0$ (see eq. (11)) gives $h_1' = -h_2'$ so we write the wall height perturbation as follows

$$h = 1 + 2h'\, e^{\lambda t}\,\sinh\sqrt{\frac{3\lambda}{\Gamma}}x \tag{30}$$

In the larger cavity, the pressure perturbation must satisfy the wave equation. Using eqs. (26) and (27), and the separation of variables $p = e^{\lambda t} P(x)$, one gets

$$\lambda^2 P = \alpha^2 P_{xx} \tag{31}$$

solution for pressure perturbation is

$$p(x,t) = e^{\lambda t}\left(a \cosh\frac{\lambda}{\alpha}(l{+}1-x) + b \sinh\frac{\lambda}{\alpha}(l{+}1-x)\right) \tag{32}$$

with a and b real coefficient. The boundary condition $p_x(x = l, t) = 0$ demands $b = 0$. Writing the continuity of pressure at $x = 1$, one obtains

$$2\Gamma\, h' \sinh\sqrt{\frac{3\lambda}{\Gamma}} = a \cosh\frac{\lambda l}{\alpha} \tag{33}$$

Finally, we require the conservation of fluid flux $\mathcal{F}$ between the two domains. The flux in the channel is given by

$$\begin{aligned}\dot{\mathcal{F}}^- &= (\bar{u}\, h)_t = -\left(\frac{p_x}{3}h^3\right)_t = -\left(\frac{\Gamma}{3}h^3 h_x\right)_t \\ &= -\lambda\sqrt{\frac{\lambda\,\Gamma}{3}}2h'\, e^{\lambda t} \cosh\sqrt{\frac{3\lambda}{\Gamma}}x + O(h'^2)\end{aligned} \tag{34}$$

where the last step is obtained by linearization of walls height perturbation. In the conduit, the rate of flux is

$$\begin{aligned}\dot{\mathcal{F}}^+ &= W\, u_t = -\frac{W}{R}p_x \\ &= \frac{\lambda}{\alpha}\frac{W}{R}e^{\lambda t} a \sinh\frac{\lambda}{\alpha}(l{+}1-x)\end{aligned} \tag{35}$$

Equating (34) and (35), one obtains the flux continuity condition

$$\sqrt{\frac{\lambda\,\Gamma}{3}}2h' \cosh\sqrt{\frac{3\lambda}{\Gamma}} + \frac{W}{\alpha\, R}a \sinh\frac{\lambda\, l}{\alpha} = 0 \tag{36}$$

Equation (36) together with eq. (33) leads to the dispersion equation

$$\sqrt{\frac{\lambda}{3\Gamma}} + \frac{W}{\alpha\, R} \tanh\frac{\lambda\, l}{\alpha} \tanh\sqrt{\frac{3\lambda}{\Gamma}} = 0 \tag{37}$$

We can see by simple inspection that no real positive part of λ can satisfy eq. (37). The simple formulation of the problem leads to a stable solution.

2.1.2 A Stability Theorem

Considering the conservation of momentum and conservation of mass equations, one can write

$$\begin{cases} u(u_t + \dfrac{1}{R}p_x) = 0 \\ \dfrac{1}{R\alpha^2}p(p_t + u_x) = 0 \end{cases} \tag{38}$$

After integrating eq. (38), one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_1^{l+1} R\, u^2 + \rho\, p\, dx &= -\,[up]_1^{l+1} \\ &= \frac{1}{\rho_1}(up)_{x=1}\end{aligned} \tag{39}$$

Conservation of flux at $x = 1$ ($\mathcal{F}^+ = Wu$ and $\mathcal{F}^- = -h^3 p_x/3$) and pressure eq. (20) provides us with

$$u = -\frac{\Gamma}{3W} h^3 h_x \tag{40}$$

or

$$up = -\frac{\Gamma^2}{3W} h^3 h_x (h-1) \tag{41}$$

As conduit wall height obeys the nonlinear diffusion equation, one gets

$$\int_0^1 h_t (h-1)\, dx = \frac{\Gamma}{3} \int_0^1 (h^3 h_x)_x (h-1)\, dx \tag{42}$$

or

$$\frac{1}{2}\frac{d}{dt} \int_0^1 (h-1)^2\, dx = \frac{\Gamma}{3} \left[h^3 h_x (h-1)\right]_0^1 - \frac{\Gamma}{3} \int_0^1 h_x^2 h^3\, dx \tag{43}$$

Collecting terms, and using $\rho_1 = 1$, one obtains

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{ \frac{\Gamma}{W} \int_0^1 (h-1)^2\, dx + \int_1^{l+1} R\, u^2 + \rho\, p\ dx \right\} \\ = -\frac{2\Gamma^2}{15W} \int_0^1 (h^{5/2})_x^2\, dx \end{aligned} \tag{44}$$

As the right-hand-size term is negative definite, the rate of change of the total energy is negative. There is only dissipation in the system and there are no growing terms possible after perturbation.

2.2 Advection From a Pressure Gradient

We abandon the assumptions made in section 2.1 and consider in this section that fluid flow is driven by a pressure gradient γ in the channel. (We note that this model corresponds to the case of an open-funnel geometry, as developed in a later section, with the limit of neglected inertia terms.) The pressure is therefore given by

$$p(x,t) = \Gamma(h-1) + \gamma\, x \tag{45}$$

where $h = 1$ is the wall height at equilibrium. The equation of continuity in the channel now reads

$$h_t = \frac{\Gamma}{3} \left(h^3 \left(h_x + \frac{\gamma}{\Gamma} \right) \right)_x \tag{46}$$

the pressure gradient introducing an effective advection term. After linearization of the perturbation solution, one obtains

$$\lambda = \frac{\Gamma}{3} m \left(m + \frac{3\gamma}{\Gamma} \right) \tag{47}$$

We write m_1 and m_2 the two solutions

$$\begin{aligned} m_1 &= -\frac{3\gamma - \sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma} \\ m_2 &= -\frac{3\gamma + \sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma} \end{aligned} \tag{48}$$

and the walls height in the channel can be written

$$h = 1 + 2\,h' \exp\left(-\frac{3\gamma}{2\Gamma}x + \lambda t\right) \sinh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma}x \tag{49}$$

In the conduit, we write the pressure perturbation such as

$$p(x,t) = \gamma + e^{\lambda t} P(x), \qquad 1 \le x \le l+1 \tag{50}$$

Solving the wave equation subjected to Neumann boundary condition at $x = l$ leads to the pressure field

$$p(x,t) = \gamma + e^{\lambda t} a \cosh \frac{\lambda}{\alpha}(l+1-x), \qquad 1 \le x \le l+1 \tag{51}$$

Writing the equation of continuity at $x = 1$, one gets

$$2\,\Gamma\,h'\,e^{-3\gamma/2\Gamma} \sinh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma} = a \cosh \frac{\lambda l}{\alpha} \tag{52}$$

We require the conservation of fluid flux $\mathcal{F}$ between the two domains. The flux rate in the channel is given by

$$\dot{\mathcal{F}}^- = (\bar{u}\,h)_t = -\left(\frac{p_x}{3}h^3\right)_t = -\frac{\Gamma}{3}\left(h^3\left(h_x + \frac{\gamma}{\Gamma}\right)\right)_t \tag{53}$$

To find the flux rate about $x = 1$, we note first that

$$\begin{aligned}
p_x &= \Gamma\,h_x + \gamma \\
h_x &= 2h' e^{\lambda t - 3\gamma/2\Gamma}\left(\frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma} \cosh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma}\right. \\
&\qquad\qquad \left. - \frac{3\gamma}{2\Gamma} \sinh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma}\right) \\
h^3 &= 1 + 6\,h' e^{\lambda t - 3\gamma/2\Gamma} \sinh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma} + O(h'^2)
\end{aligned} \tag{54}$$

after some algebra, one obtains

$$\begin{aligned}
\dot{\mathcal{F}}^- = -\frac{2\Gamma}{3}\,\lambda h' &\left[\frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma} \cosh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma}\right. \\
&\quad \left. + \frac{3\gamma}{2\Gamma} \sinh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma}\right] e^{\lambda t - 3\gamma/2\Gamma}
\end{aligned} \tag{55}$$

The flux rate in the conduit does not differ from eq. (35). Equating (35) and (55), one obtains the dispersion relation

$$\begin{aligned}
\left(\frac{W}{\alpha R} \tanh \frac{\lambda l}{\alpha} + \frac{\gamma}{2\Gamma}\right) \tanh \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{2\Gamma}& \\
+ \frac{\sqrt{9\gamma^2 + 12\lambda\Gamma}}{6\Gamma} &= 0
\end{aligned} \tag{56}$$

The leftmost quantity in parenthesis must be negative in order to a real positive solution for λ to exist. Also notice that eq. (56) simplifies to eq. (37) in the case $\gamma = 0$. Fig. 2 shows the norm of dispersion relation of eq. (56) on the complex plane with $W/\rho_1 c = 1/2$, $\Gamma = 1$ and $\gamma = -2$. The black circles in Fig. 2 correspond to the natural pulsation of the sound waves in the conduit. The zeroes of the dispersion equation -corresponding to blue circles in the contour plot- are close to the frequency corresponding to waves bouncing back and forth in the conduit. The associated eigenvalues are located on the real negative axis and correspond to decaying oscillation modes.

2.2.1 Stability Condition

Integrating the governing equations in the two domains and applying conservation of flux at $x = 1$, one finds

$$
\begin{aligned}
\frac{d}{dt}\left\{ \frac{\Gamma}{W}\frac{1}{2}\int_0^1 (h - H + \frac{\gamma}{\Gamma})^2\,dx + \frac{1}{2}\int_1^{l+1} R\,u^2 + \rho\,p\,dx \right\}& \\
= -\frac{\Gamma}{W}\int_0^1 h^3 h_x (h_x + \frac{\gamma}{\Gamma})\,dx& \\
= -\frac{\Gamma}{W}\left(\frac{2}{5}\int_0^1 (h^{5/2})_x^2\,dx + \frac{\gamma}{4\Gamma}\left[h^4\right]_0^1 \right)&
\end{aligned} \tag{57}
$$

For negative pressure gradient ($\gamma < 0$), any perturbation remains stable for $h(1) < h(0)$.

3 Fluid Flow and Sound-Wave Resonator

We now adopt a more realistic formulation of the problem where the constrained pressure gradient in the channel driving the fluid flow is compatible with the elastic deformation of the channel walls. At static equilibrium, the channel closes itself to accommodate the pressure difference between inlet and outlet. By writing the relation between pressure and wall height in the channel as follows

$$p(x,t) = \Gamma\,h \tag{58}$$

and using eqs. (19) and (58), the wall height satisfies the nonlinear diffusion equation

$$h_t = \frac{\Gamma}{3}\left(h^3 h_x\right)_x \tag{59}$$

subjected to the boundary conditions $p(0,t) = p_0$ and $p(1,t) = p_1 \equiv p_0 + \gamma$ at equilibrium, where γ is an effective pressure gradient. The walls height solving eqs. (19) and (58) at equilibrium ($h_t = 0$) with above-mentioned boundary conditions is

$$h(x) \equiv H(x) = \frac{1}{\Gamma}\left(\left(p_1^4 - p_0^4\right)x + p_0^4\right)^{1/4} \tag{60}$$

where we defined $H(x)$ to refer to the equilibrium, non-perturbed, solution. We now perform linearization about equilibrium as follows

$$
\begin{aligned}
h(x,t) &= H(x) + e^{\lambda t} h'(x), \qquad 0 \le x \le 1 \\
p(x,t) &= p_1 + e^{\lambda t} P(x), \qquad 1 \le x \le l+1
\end{aligned} \tag{61}
$$

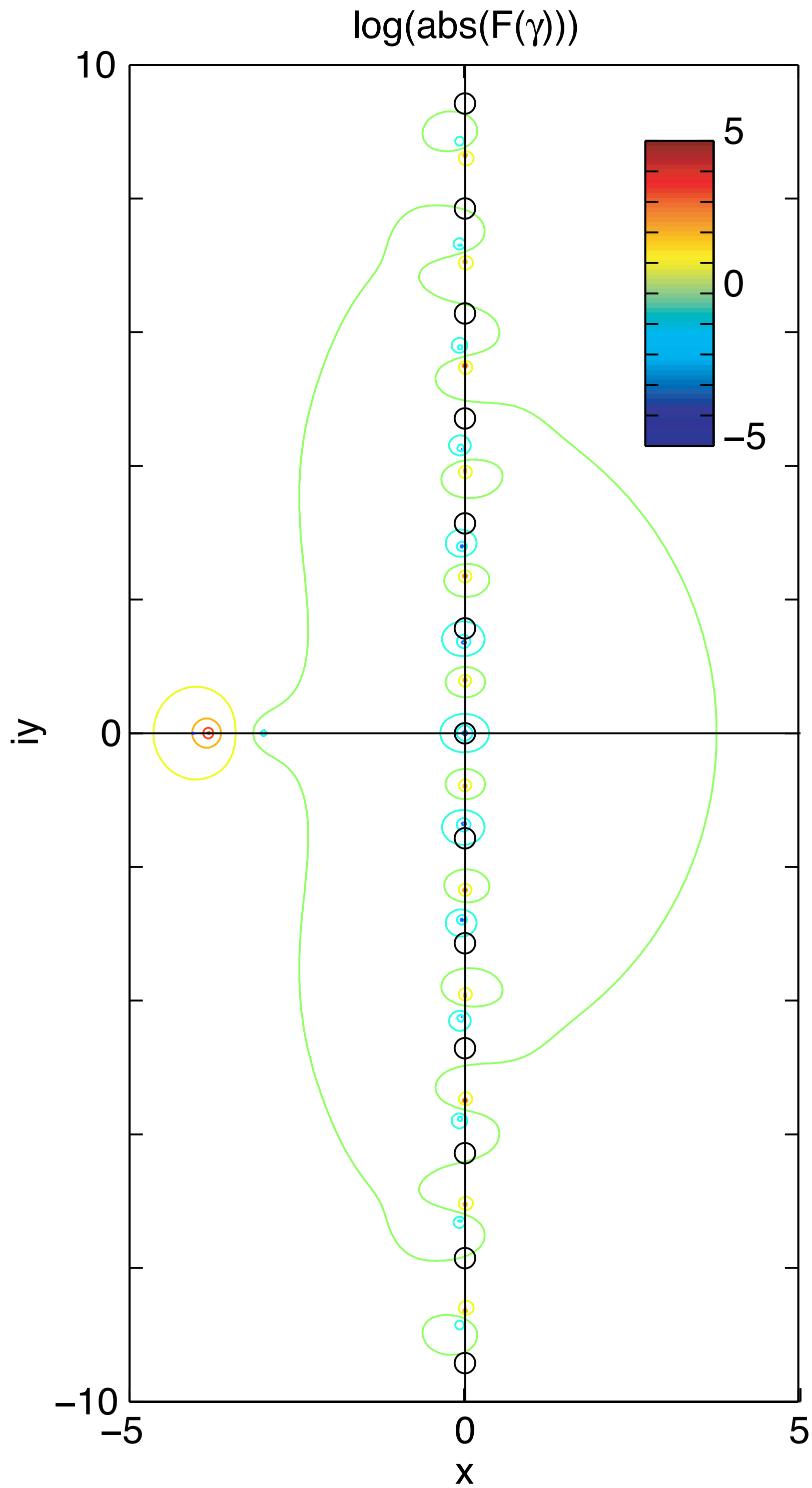

Figure 2: Contour plot on the complex plane of the dispersion relation of eq. (56) with $W/\rho_1 c = 1/2$, $\Gamma = 1$ and $\gamma = -2$. Black circles correspond to the natural notes of the conduit. There are no growing oscillatory modes.

where $h'(x)$ is small and unknown. We first note the few following results

$$
\begin{aligned}
h^3 &= H^3 + 3H^2 e^{\lambda t} h' + O(h'^2) \\
h^3 h_x &= H^3 H_x + \left(H^3 h'_x + 3H_x H^2 h'\right) e^{\lambda t} + O(h'^2) \\
(h^3 h_x)_x &= \left(H^3\, h'_{xx} + 6H_x H^2\, h'_x - 3H_x^2 H\, h'\right) e^{\lambda t} h' + O(h'^2)
\end{aligned}
\tag{62}
$$

where we used the fact that H satisfies, by definition,

$$
(H^3 H_x)_x = 3H_x^2 H^2 + H^3 H_{xx} = 0 \tag{63}
$$

Expanding the perturbed solution in the nonlinear diffusion equation one obtains an ordinary differential for $h'(x)$,

$$
\frac{3}{\Gamma}\lambda\, h' = H^3\, h'_{xx} + 6H_x H^2\, h'_x - 3H_x^2 H\, h' \tag{64}
$$

or simply stated,

$$
\mathcal{L}[h'] = \lambda\, h' \tag{65}
$$

where the relevant boundary conditions for the eigenvalue problem are $h'(0,t) = 0$ and continuity of pressure and mass flux at $x = 1$. The flux rate at $x = 1^+$, in the larger cavity is

$$
\dot{\mathcal{F}}^+ = u_t W = -\frac{1}{R} p_x W = a\lambda\, \frac{W}{\alpha R} e^{\lambda t} \sinh \frac{\lambda l}{c} \tag{66}
$$

In the constriction, at $x = 1^-$, the flux rate is given by

$$
\begin{aligned}
\dot{\mathcal{F}}^- &= (\bar{u}h)_t = -\frac{1}{3}\left(h^3 p_x\right)_t = -\frac{\Gamma}{3}\left(h^3 h_x\right)_t \\
&= -\frac{\Gamma}{3}\lambda\, e^{\lambda t}\left[H^3 h'_x + 3H_x H^2 h'\right] + O(h'^2)
\end{aligned}
\tag{67}
$$

Continuity of flux at $x = 1$ gives at the first order

$$
a\, \frac{W}{\alpha R} \sinh \frac{\lambda l}{\alpha} + \frac{\Gamma}{3}\left(H^3 h'_x + 3H_x H^2 h'\right) = 0 \tag{68}
$$

We insist on constraining continuity of pressure at $x = 1$. We have

$$
p(1^+, t) = p_1 + a\, e^{\lambda t} \cosh \frac{\lambda l}{c} \tag{69}
$$

and

$$
p(1^-, t) = \Gamma h = p_1 + \Gamma\, e^{\lambda t} h' \tag{70}
$$

so continuity of pressure at $x = 1$ gives us

$$
\Gamma h' = a\, \cosh \frac{\lambda l}{c} \tag{71}
$$

Combining the two constraints at $x = 1$, one obtains

$$
h' \frac{W}{\alpha R} \tanh \frac{\lambda l}{\alpha} + \frac{1}{3}\left(H^3 h'_x + 3H_x H^2 h'\right) = 0 \tag{72}
$$

Numerical inspection of solutions of the dispersion equation of eq. (64) subjected to eq. (72) at $x = 1$ reveals only solutions with negative real parts, such as the one shown in Fig. 3.

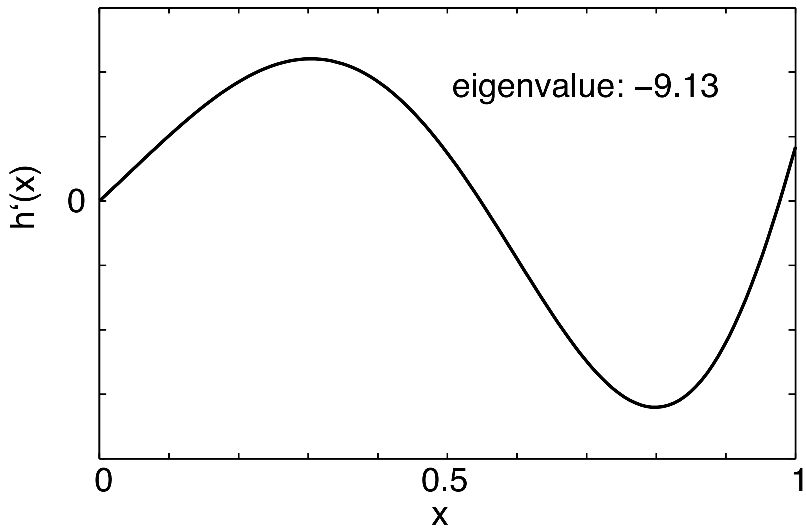

Figure 3: An eigenfunction of the dispersion function of eq. (64). Solutions correspond to decaying modes only, with negative real part eigenvalues.

3.1 Inertia Terms

We now consider the case where inertia terms in Navier-Stokes equation are small but not negligible. The governing equations adequate for lubrication theory become

$$\begin{aligned} R\,(u_t + u\,u_x + v\,u_y) &= -p_x + u_{yy} \\ p_y &= 0 \end{aligned} \tag{73}$$

subjected to the boundary conditions $u(x,h,t) = u_y(x,0,t) = 0$. At leading order, when derivatives in the x-direction and acceleration can be neglected, the solution can be written

$$u(y,t) = \frac{3}{2}U\,\frac{h^2 - y^2}{h^2} \tag{74}$$

where $h = h(x,t)$ and by definition,

$$U = \frac{1}{h}\int_0^h u\,dy\ . \tag{75}$$

We assume that small inertia terms only perturb the above steady solution so that solution of eq. (73) can be of the form

$$u(x,y,t) = \frac{3}{2}U(x,t)\,\frac{h^2 - y^2}{h^2} \tag{76}$$

However, the eq. (76) does not provide an accurate way to evaluate u_{yy} so, when integrating eq. (73), we look for a certain projection f that allows us to avoid terms of the form u_{yy}

$$\begin{aligned} \int_0^h f(y)\,u_{yy}\,dy &= [f(y)\,u_y]_0^h - \int_0^h f'(y)u_y\,dy \\ &= [f(y)\,u_y]_0^h - \left[f'(y)\,u\right]_0^h + \int_0^h f''(y)u\,dy \end{aligned} \tag{77}$$

Setting $f''(y) = \text{cste}$ and $f(h) = f'(0) = 0$, one gets

$$\begin{aligned} f(x,y,t) &= 1 - y^2/h^2 \\ \int_0^h f u_{yy}\, dy &= -\frac{2}{h^2}\int_0^h u\, dy = -2U/h \end{aligned} \tag{78}$$

Notice that the projection kernel $f(y)$ is of the form of the steady solution of eq. (74). Projecting the lubrication approximation of the Navier-Stokes equations on $f(x,y,t)$, we look for a solution for $h(x,t)$ and $U(x,t)$. We first notice, using $u_x + v_y = 0$ and $v(0) = 0$ that one can write

$$v(x,y,t) = -\frac{3}{2}U_x y\left(1 - \frac{y^2}{3h^2}\right) - U h_x \frac{y^3}{h^3} \tag{79}$$

The projected terms are

$$\begin{aligned} \int_0^h f u_t\, dy &= \frac{4}{5}U_t h + \frac{2}{5}h_t U \\ \int_0^h f u_{xx}\, dy &= \frac{2}{5}\left[2U_{xx}h + 2U_x h_x + U(h_{xx} - 3h_x^2/h)\right] \\ \int_0^h f u_{yy}\, dy &= -2U/h \\ \int_0^h f u u_x\, dy &= \frac{36}{35}U_x U h + \frac{12}{35}U^2 h_x \\ \int_0^h f v u_y\, dy &= \frac{6}{35}U\left[3U_x h + U h_x\right] \\ \int_0^h -f\frac{p_x}{\rho}\, dy &= -\frac{2}{3}h\frac{p_x}{\rho} \end{aligned} \tag{80}$$

Defining the flux,

$$q = \int_0^h u\, dy = h\,U \tag{81}$$

the conservation of mass can be written

$$h_t + q_x = 0 \tag{82}$$

and collecting terms, one gets the coupled equations

$$\begin{cases} R\left(q_t + \dfrac{17}{7}\dfrac{q_x q}{h} - \dfrac{9}{7}\dfrac{q^2 h_x}{h^2}\right) = -\dfrac{5}{6}p_x h - \dfrac{5q}{2h^2} \\ h_t + q_x = 0 \end{cases} \tag{83}$$

Using the stress-strain relation

$$p(x,t) = \Gamma\, h(x,t) \tag{84}$$

one obtains the governing coupled equations

$$\begin{cases} R\left(q_t h^2 - \dfrac{17}{7}h_t q h - \dfrac{9}{7}q^2 h_x\right) = -\dfrac{5}{6}\Gamma h_x h^3 - \dfrac{5q}{2} \\ h_t + q_x = 0 \end{cases} \tag{85}$$

3.1.1 Steady-State Solution

We look for a steady solution for the wall height $H(x)$ and flux Q. Dropping time derivatives in eq. (85) and using the fact that at steady state $H_t = Q_x = 0$ and that Q is constant, one has

$$H_x \left(\frac{5}{6} \Gamma H^3 - \frac{9}{7} R\, Q^2 \right) + \frac{5}{2} Q = 0 \tag{86}$$

Integrating, we have

$$\frac{1}{12} \Gamma (H^4 - H_0^4) - \frac{18}{35} R\, Q^2 (H - H_0) + Q\, x = 0 \tag{87}$$

If $R \ll 1$, and the flow is constrained by the pressure p_0 at $x = 0$ and p_1 at $x = 1$, one has

$$\begin{aligned} Q &= \frac{p_0^4 - p_1^4}{12 \Gamma^3} \\ H &= \frac{1}{\Gamma} \left((p_1^4 - p_0^4) x + p_0^4 \right)^{1/4} \end{aligned} \tag{88}$$

which correspond to solution of eq. (60), obtained for low Reynolds number. For finite Reynolds number, the steady-state flux is

$$Q = \frac{35 - \sqrt{1225 + 210\, \Gamma (H_1^4 - H_0^4)(H_1 - H_0)}}{36 R\, (H_1 - H_0)} \tag{89}$$

3.1.2 Linear Stability Analysis

We consider perturbations of the wall height, flux and pressure of the form

$$\begin{aligned} h(x,t) &= H(x) + e^{\lambda t} h'(x) \\ q &= Q + e^{\lambda t} q'(x) \\ p &= p_1 + a\, e^{\lambda t} \cosh \frac{\lambda}{c} (l + 1 - x) \end{aligned} \tag{90}$$

where perturbations $h'(x)$ and $q'(x)$ are supposed small. Inserting eq. (90) into eq. (85) and neglecting powers of h' and q', one obtains

$$\begin{aligned} R \left(\lambda q' H^2 - \frac{17}{7} \lambda h' H Q - \frac{9}{7} (h'_x Q^2 + 2 q' H_x) \right) \\ = -\frac{5}{6} \Gamma \left(H^3 h'_x + 3 H^2 H_x h' \right) - \frac{5}{2} q' \end{aligned} \tag{91}$$

The relevant boundary conditions are $h'(0,t) = q'(0,t) = 0$, continuity of pressure and flux at $x = 1$. The continuity of pressure gives

$$\Gamma\, h' = a \cosh \frac{\lambda l}{c} \tag{92}$$

Flux perturbation on the 1^+ side is

$$\begin{aligned} \dot{\mathcal{F}}^+ &= W\, u_t = -\frac{W}{\rho_1} p_x \\ &= a \lambda \frac{W}{\rho_1 c} e^{\lambda t} \sinh \frac{\lambda l}{c} \end{aligned} \tag{93}$$

On the other side, the flux perturbation is simply

$$\dot{\mathcal{F}}^{-} = \lambda\, e^{\lambda t} q' \tag{94}$$

Equating the fluxes on both sides, we get the boundary condition

$$q' - h' \frac{\Gamma W}{\rho_1 c} \tanh \frac{\lambda l}{c} = 0 \tag{95}$$

We explore the solutions of the corresponding eigenvalue problem numerically. We find a set a growing oscillatory modes associated with complex eigenvalues with positive real parts. The corresponding eigenfunctions are shown in Fig. 4.

3.2 Open-Funnel Walls

We consider now the case where the wall in the channel opens in the absence of pressure. Before fluids flow in the channel, the wall height has the equilibrium value $H^e(x)$ (see Fig. 5). And the stress-strain relation becomes

$$p(x,t) = \Gamma\left(h(x,t) - H^e(x)\right) \tag{96}$$

Using eq. (83) at equilibrium, where $h = H(x)$ and $q = Q$, we have

$$\begin{cases} R\frac{9}{7}Q^2 H_x = \frac{5}{6}\Gamma\left(H_x - H_x^e\right) H^3 + \frac{5}{2}Q \\ H_t = Q_x = 0 \end{cases} \tag{97}$$

For a general form of $H^e(x)$, the latter has no trivial solutions. We consider a special case where H_x^e is such that $H_x(x) = 0$. Under this assumption, one has $H = Q = 1$ and

$$H_x^e = \frac{3}{\Gamma} \tag{98}$$

Without inertia ($R = 0$), the governing equation (83) becomes

$$h_t = -\frac{\Gamma}{3}\left(h^3\left(h_x - H_x^e\right)\right)_x \tag{99}$$

which is similar to the formulation of section 2.2 if we write

$$\gamma = -\Gamma H_x^e \tag{100}$$

The physical grounds of the somewhat artificial pressure γ introduced in section 2.2 are the presence of opening walls in the conduit in the absence of pressure. Inspection of the dispersion relation in this case did not reveal real-positive eigenvalues so we proceed with inertia terms.

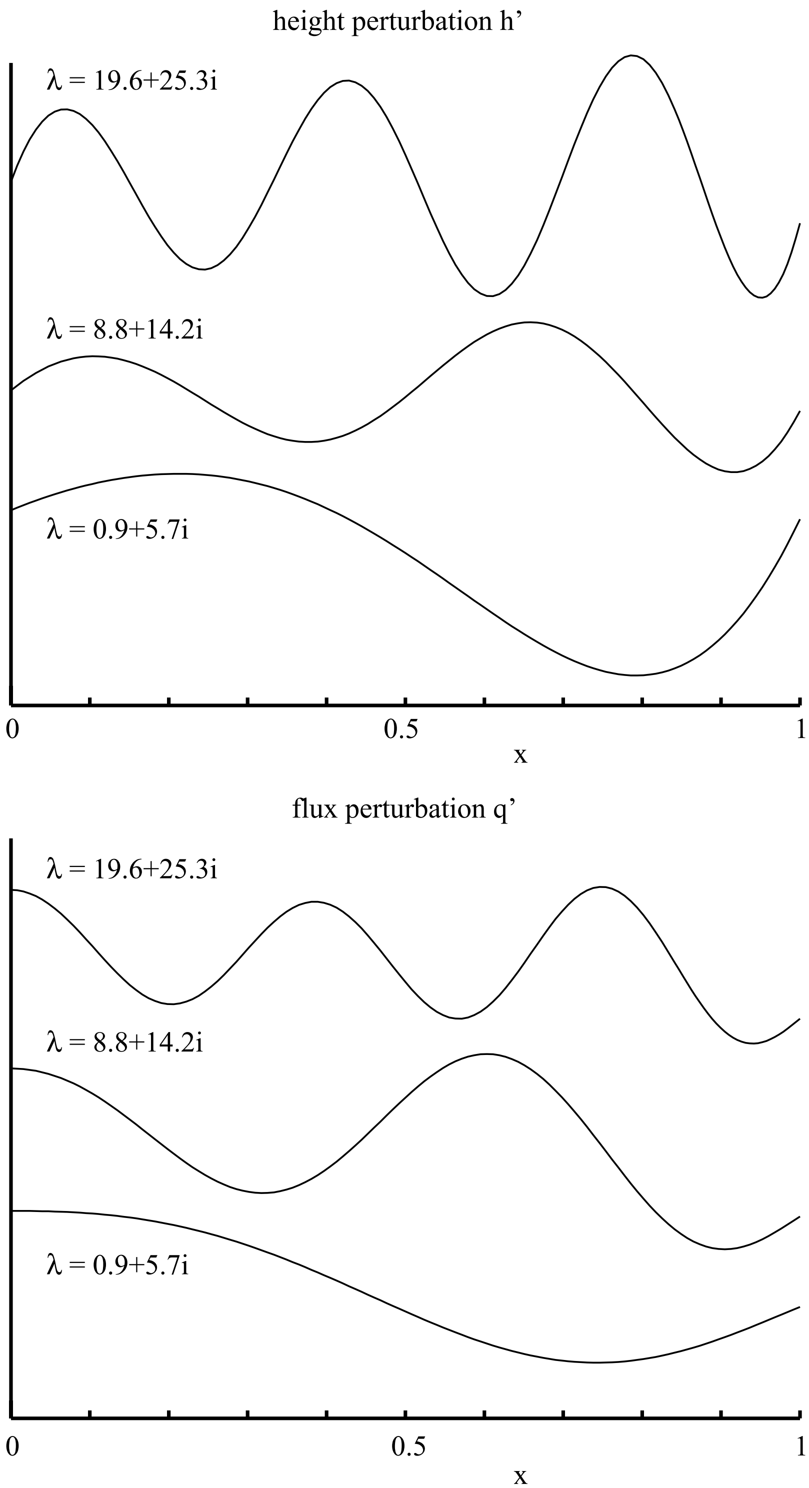

Figure 4: The first three growing eigenmodes of the system with associated eigenvalues. Top panel show the height eigenfunction $h'(x;\lambda)$ and bottom panel shows the flux eigenfunctions $q'(x;\lambda)$ in the channel.

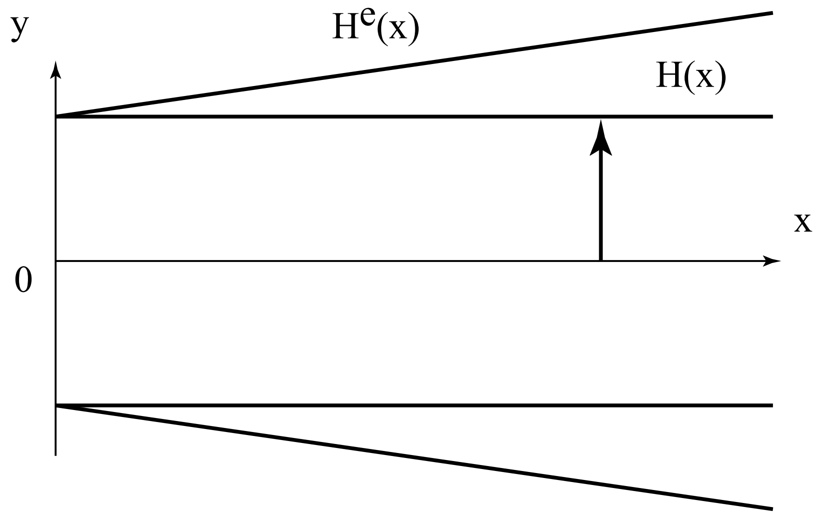

Figure 5: In the absence of fluid flow, the walls have the height $H^e(x)$. The latter is chosen such as the height of the walls is uniform when at equilibrium with steady fluid flow.

3.2.1 Stability Analysis

We write the perturbation as follows

$$\begin{aligned} h(x,t) &= 1 + h'(x)\, e^{\lambda t} \\ q(x,t) &= 1 + q'(x)\, e^{\lambda t} \\ p(x,t) &= \Gamma\,(1 - H^e(1)) + p'\, e^{\lambda t} \end{aligned} \tag{101}$$

Introducing the latter decomposition into eq. (83), one finds the system of coupled equations

$$\begin{cases} R\left(\lambda q' + \dfrac{17}{7} q'_x - \dfrac{9}{7} h'_x\right) = -\dfrac{5}{6}\Gamma h'_x - \dfrac{5}{2} q' + \dfrac{15}{2} h' \\ \lambda h' + q'_x = 0 \end{cases} \tag{102}$$

This result is similar to previous dispersion relations except for the newly appearing forcing term in h' on the right-hand side, so we expect to find emergent solutions, if any, for high enough Reynolds number R. The pressure continuity condition provides us with the condition

$$\Gamma\, h' \;=\; a\, \cosh \frac{\lambda l}{\alpha} \tag{103}$$

The continuity of flux at $x = 1$ leads to the second constraint

$$q' = a \frac{W}{\alpha R} \sinh \frac{\lambda l}{\alpha} \tag{104}$$

Combining the two, we obtain the boundary condition at $x = 1$

$$\lambda q' + \Gamma\, q'_x\, \frac{W}{\alpha R} \tanh \frac{\lambda l}{\alpha} = 0 \tag{105}$$

other relevant conditions are simply $h'(0) = q'_x(0) = 0$. The flux perturbation obeys the second-order homogeneous ordinary differential equation

$$\begin{aligned} \left(\frac{9}{7}R - \frac{5}{6}\Gamma\right) q'_{xx} + \left(\frac{17}{7}\lambda R + \frac{15}{2}\right) q'_x \\ + \lambda\left(\lambda R + \frac{5}{2}\right) q' = 0 \end{aligned} \tag{106}$$

The determinant of the characteristic polynomial is

$$\Delta = \left(\frac{17}{7}\lambda R + \frac{15}{2}\right)^2 - 4\lambda\left(\frac{9}{7}R - \frac{5}{6}\Gamma\right)\left(\lambda R + \frac{5}{2}\right) \tag{107}$$

Using the boundary condition $q'_x(0) = 0$, the solution flux perturbation can be written

$$q'(x) = e^{A\,x}\ (A\,\sinh(B\,x) - B\,\cosh(B\,x)) \tag{108}$$

where A and B are

$$\begin{aligned} A &= -\frac{1}{2}\left(\frac{17}{7}\lambda R + \frac{15}{2}\right)\Big/\left(\frac{9}{7}R - \frac{5}{6}\Gamma\right) \\ B &= \sqrt{\Delta}\Big/\left(\frac{18}{7}R - \frac{10}{6}\Gamma\right) \\ C &= \left(A^2 - B^2\right)/\lambda = \left(\lambda\,R + \frac{5}{2}\right)\Big/\left(\frac{9}{7}R - \frac{5}{6}\,\Gamma\right) \end{aligned} \tag{109}$$

Using boundary condition of eq. (105) at $x = 1$, one obtains the dispersion relation

$$\left(A + \frac{\Gamma W}{\alpha R}\,C\tanh\frac{\lambda l}{\alpha}\right)\tanh B - B = 0 \tag{110}$$

Fig. 6 shows a view of the dispersion equation as a function of λ on the complex plane for the parameters $\Gamma = 1$, $R = 1$, $l/\alpha = 1$ and $W/\alpha = 2$.

4 Conclusions

We investigated the potential of flow of an incompressible fluid in a channel with elastic walls coupled to a sound wave resonator to generate self-generated growing instabilities. We derived a simplified version of the Navier-Stokes equations valid for the small aspect ratio of the thin channel. We obtain equations similar to the lubrication theory, but with the inertia terms conserved. We found that simple models without flow (zero Reynolds number) did not generate instabilities. The frequency of corresponding decaying oscillation modes are close to the normal modes of sound waves in the conduit. We found linear instability corresponding to growing oscillatory modes for finite yet small Reynolds number. The small Reynolds number required for flow instability favors the occurrence of tremors in various environments, as constraints on minimum flow speed or fluid viscosity (quantities linearly related to the Reynolds number) are relaxed. The growing modes corresponding to low

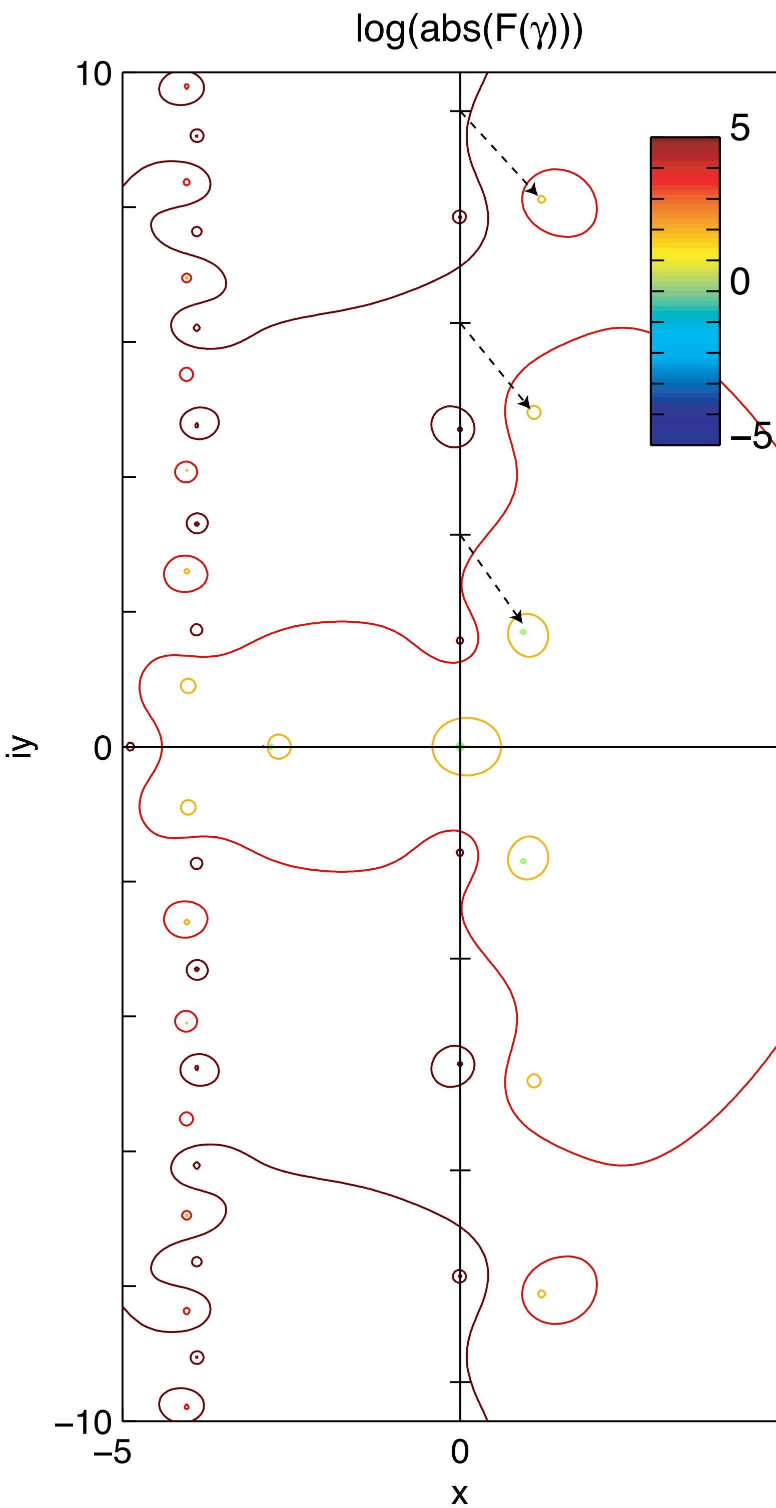

Figure 6: Map of the dispersion function as a function of eigenvalue λ on the complex plane. The zeroes indicate the position of the eigenvalues. The arrow indicate the migration of the zeroes from their position at zero Reynolds number, close to the natural notes of the conduit.

Reynolds number are associated to long time scales, giving possibly rise to long period events. The frequency of growing oscillations are smaller than the normal modes of the conduit. The smaller period of growing oscillations compare to the natural harmonics of sound wave in the conduit relaxes constraints on the linear inferred dimension of the body hosting the tremor source (water reservoir or magma chamber) and may explain the source of frequency drift observed in tremors. The proposed mechanism consisting of pressure perturbation of the fluid flow by pressure (sound) waves may be responsible for the sustained activity of volcanic tremors. The role of compressibility might affect the apparent frequency of growing modes in the channel. Implications of flow of a compressible fluid need to be investigated further as the presence of compressible fluids are relevant to the occurrence of tremors at shallow depth in volcanic plumbing systems. Our suggested mechanism may be relevant to long-period or tremor events in both magmatic and aquifer environment.

Appendix - Compressible Fluid Flow

Governing Equations

Under the assumptions adequate for lubrication theory, the Navier-Stokes equation for compressible flow reduces to

$$\begin{aligned} &R(\rho\, u_t + \rho\, u u_x + \rho\, v u_y) = -p_x + \eta\, u_{yy} \\ &p_y = \rho_y = 0 \end{aligned} \tag{111}$$

where η is the kinematic viscosity and ρ is the fluid density. The momentum equation is accompanied by the mass conservation equation

$$\rho_t + \nabla \cdot (\rho\, v) = 0 \tag{112}$$

where the density ρ is related to the pressure by the (non-dimensionalized) equation of state

$$p = \rho^{\gamma} \tag{113}$$

Approximations of the Navier-Stokes equations in section 2 are still valid for compressible flow and we have $p_y = \rho_y = 0$. Defining the volume flux

$$q = \int_0^h u\, dy \tag{114}$$

the conservation of mass can be written

$$(\rho h)_t + (\rho q)_x = 0 \tag{115}$$

At small Reynolds number, one has

$$q = -\frac{1}{3} p_x\, h^3 \tag{116}$$

and using the relationship between wall height and pressure $p = \Gamma h$, one obtains the non-linear diffusion equation

$$\left(h^{\frac{\gamma+1}{\gamma}}\right)_t = \frac{\Gamma}{3}\left(h_x h^{\frac{3\gamma+1}{\gamma}}\right)_x \tag{117}$$

At equilibrium, when $h_t = 0$, one has

$$p = \left[\left(\rho_1\, p_1^4 - \rho_0\, p_0^4\right) x + \rho_0\, p_0^4\right]^{\frac{\gamma}{4\gamma+1}} \tag{118}$$

where H_0, H_1 and p_0, p_1 are the wall height and pressure respectively, at $x = 0$ and $x = 1$, respectively.

Small Reynolds Number

We evaluate the linear stability of the problem sketched in Fig. 1, with compressible flow and small Reynolds number. We write the perturbation as

$$\begin{aligned} h &= H(x) + e^{\lambda t} h'(x), \qquad && 0 \le x \le 1 \\ p &= p_1 + e^{\lambda t} p'(x), && 1 \le x \le l+1 \end{aligned} \tag{119}$$

subjected to the boundary conditions $h'(0) = p_x(l+1) = 0$ and conservation of pressure and flux at $x = 1$. The same fluid flow in the larger channel, and there the pressure p varies about the level p_1 at the exit of the contrition, so we have

$$p = p_1 + c^2 \rho \tag{120}$$

where

$$c^2 \equiv \gamma p_1^{\frac{\gamma-1}{\gamma}} \tag{121}$$

In the larger channel, the pressure obeys the wave equation subjected to $p_x(l+1) = 0$ and the solution is

$$p(x,t) = p_1 + e^{\lambda t}\, a\, \cosh \frac{\lambda}{c}(l{+}1 - x) \tag{122}$$

Expanding the decomposition of eq. (119) into the nonlinear diffusion equation of eq. (117) and keeping the linear terms in h', one finds

$$\frac{3}{\Gamma}\lambda \frac{\gamma+1}{\gamma} h' = \frac{3\gamma+1}{\gamma}\left[2 H_x H^2 h'_x - H_x^2 H\, h'\right] + H^3 h'_{xx} \tag{123}$$

Notice that the latter reduces to eq. (64) for the limiting case $\gamma \to \infty$, i.e., incompressible flow. The conservation of flux at $x = 1$ is $[\rho\, q]_{1^-}^{1^+} = 0$. The mass flux rates are

$$\dot{\mathcal{F}}^{+} = (\rho_1 W u_t) = a\lambda \frac{W}{c}\, e^{\lambda t}\, \sinh \frac{\lambda l}{c} \tag{124}$$

and

$$\begin{aligned} \dot{\mathcal{F}}^{-} &= (\rho\, q)_t = -\frac{1}{3}\left[\rho p_x h^3\right]_t = -\frac{1}{3}\Gamma^{\frac{\gamma+1}{\gamma}}\left[h_x h^{\frac{3\gamma+1}{\gamma}}\right]_t \\ &= -\lambda \frac{1}{3}\Gamma^{\frac{\gamma+1}{\gamma}}\left[\frac{3\gamma+1}{\gamma} H_x H^{\frac{2\gamma+1}{\gamma}} h' + H^{\frac{3\gamma+1}{\gamma}} h'_x\right] e^{\lambda t} \end{aligned} \tag{125}$$

The conservation of pressure at $x = 1$ gives

$$\Gamma\, h' = a\, \cosh \frac{\lambda l}{c} \tag{126}$$

Using $\rho_1 = \Gamma^{\frac{1}{\gamma}} H^{\frac{1}{\gamma}}$ at $x = 1$, we obtain the boundary condition

$$h' \frac{W}{\rho_1 c} \tanh \frac{\lambda l}{c} + \frac{1}{3} \left[\frac{3\gamma + 1}{\gamma} H_x H^2 h' + H^3 h'_x \right] = 0 \quad (127)$$

Examination of numerical solutions of the eigenvalue problem does not reveal any eigenvalue with positive real parts.

Finite Inertial Terms

We consider the case where fluid acceleration is non-negligible. In the context of lubrication theory, the governing equations are

$$\begin{aligned} & R(u_t + u u_x + v u_y) = -p_x + u_{yy} \\ & p_y = \rho_y = -0 \\ & \rho_t + (\rho u)_x + (\rho v)_y = 0 \end{aligned} \quad (128)$$

We write the solution u as follows

$$u(x, y, t) = \frac{3}{2} U(x, t) \left(1 - \frac{y^2}{h^2} \right) \quad (129)$$

Integration of the conservation-of-mass equation gives

$$\rho\, v = -\rho_t y - \frac{3}{2} (\rho U)_x\, y \left(1 - \frac{y^2}{3h^2} \right) - \rho U h_x \frac{y^3}{h^3} \quad (130)$$

We now project the governing equation on $f = 1 - y^2/h^2$, in particular

$$\begin{aligned} \int_0^h f \rho\, v\, u_y\, dy &= \frac{6}{35} U \left[3\, \rho\, U_x h + \rho\, U h_x + 3 \rho_x U h + \frac{7}{3} h \rho_t \right] \\ &= -\frac{6}{35} \frac{q}{h} \left[3(\rho h)_t + 2 \rho q \frac{h_x}{h} - \frac{7}{3} h \rho_t \right] \end{aligned} \quad (131)$$

where the last line was obtained using conservation of mass. Other terms of the governing equation are otherwise identical to those in eq. (80). After some algebra, we obtain the system of coupled equations

$$\begin{cases} R \left((\rho\, q)_t + \frac{17}{7} \frac{q}{h} (\rho\, q)_x - \frac{9}{7} \frac{q^2}{h^2} (\rho\, h)_x \right) = -\frac{5}{6} p_x h - \frac{5}{2} \frac{q}{h^2} \\ (\rho\, h)_t + (\rho\, q)_x = 0 \end{cases} \quad (132)$$

Using the dimension-less equation of state $p = \rho^\gamma$ and the continuity condition $p = \Gamma\, h$, we obtain the set of coupled equations

$$\begin{cases} R \left(\left(h^{\frac{1}{\gamma}} q \right)_t h^2 + \frac{17}{7} q h \left(h^{\frac{1}{\gamma}} q \right)_x - \frac{9}{7} q^2 \left(h^{\frac{\gamma+1}{\gamma}} \right)_x \right) \\ \qquad = -\frac{5}{6} \Gamma h_x h^3 - \frac{5}{2} q \\ \left(h^{\frac{\gamma+1}{\gamma}} \right)_t + \left(h^{\frac{1}{\gamma}} q \right)_x = 0 \end{cases} \quad (133)$$

The set of equations reduces to eq. (83) in the limit $\gamma \to \infty$.

Steady-State Solution

The expression for the wall height at steady state $H(x)$ can be obtained by noticing that $\left(H^{\frac{1}{\gamma}} q\right)_x = 0$. We obtain the following

$$\frac{1}{12}\Gamma\left(H^4 - H_0^4\right) - \frac{18}{35}Q^2 R\left(H^{\frac{\gamma+1}{\gamma}} - H_0^{\frac{\gamma+1}{\gamma}}\right) - Q\,x = 0 \tag{134}$$

The flux Q compatible with boundary conditions on H at $x = 0$ and $x = 1$ is given by

$$Q = \frac{-35 + \left(1225 + 210R\Gamma\left(H_1^4 - H_0^4\right)\left(H_1^{\frac{\gamma+1}{\gamma}} - H_0^{\frac{\gamma+1}{\gamma}}\right)\right)^{1/2}}{36\,R\left(H_1^{\frac{\gamma+1}{\gamma}} - H_0^{\frac{\gamma+1}{\gamma}}\right)} \tag{135}$$

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