

The Stokes drift of internal gravity wave groups

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1 Introduction

When periodic surface gravity waves propagate they have associated with them a wave-induced mean flow known as the Stokes drift after George Gabriel Stokes who first derived a theoretical description for this drift in 1847. For periodic surface gravity waves, the drift manifests itself as a net horizontal velocity of a Lagrangian particle that remains after the periodic part of the motion has been averaged out. The latter motion corresponds to elliptic particle trajectories when the wave length associated with the periodic motion is of comparable magnitude to the water depth (intermediate depth) and to circular particle trajectories when the depth is large compared to the wave length (so-called deep water waves). As a result of the drift, these orbits do not close perfectly, as illustrated by a few orbits in figure 1. Multiplied by the density of water, the Stokes drift of periodic surface gravity waves is associated with wave momentum. Taking a Lagrangian approach, Stokes (1847) found the following expression for the horizontal drift velocity at the free surface for regular waves in deep water and to second order of approximation:

$$\bar{u} = \frac{1}{T} \int_0^T u(x(t), \eta(t), t) dt = c(ka)^2 + O((ka)^4), \quad (1)$$

where $c = \sqrt{g/k}$ is the wave celerity or phase speed, g is the gravitational constant, $k = 2\pi/\lambda$ is the wave number of the regular wave, and a its linear amplitude. Stokes drift is thus a non-linear phenomenon (cf. $(ka)^2$) that can be derived from linear theory. Stokes (1847) considered both the effect of horizontal displacement $x(t) = x_0 + \Delta x(t)$ and vertical displacement at the free surface $z(t) = \xi(t)$ on the drift velocity (1), equal to the displacement per wave period T . Compatible results were first derived in an Eulerian framework by Starr (1947).

For packets or groups of surface gravity waves, the Stokes drift near the surface and its associated momentum in the direction of propagation of the packet is balanced by a return flow at depth below the surface in the opposing direction and of opposite sign and equal magnitude, as first shown by Longuet-Higgins & Stewart (1962). Their derivation is reproduced in appendix A for completeness and the streamlines of the return flow are shown in

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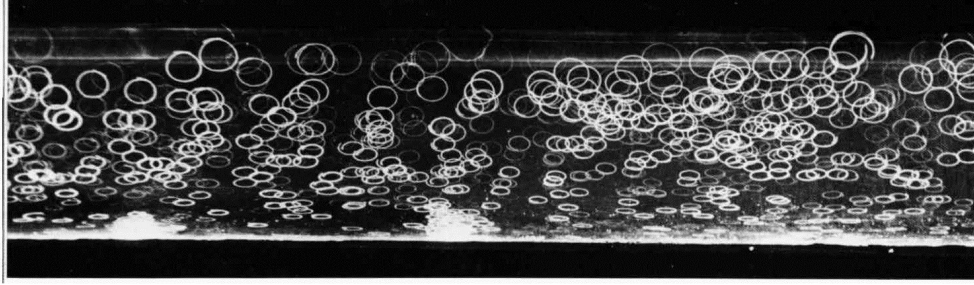


Figure 1: Particle trajectories underneath two-dimensional periodic surface gravity waves from Wallet & Ruellan (1950) reproduced in Van Dyke (1982). The waves are only moderately non-linear and the net horizontal drift is only visually apparent for a few orbits.

figure 2. Combining the Stokes drift and the return flow, leads to a zero vertically-integrated momentum associated with the travelling wave packet as emphasized by McIntyre (1981). In the terminology of the derivation of Longuet-Higgins & Stewart (1962), the different wave number components that make up the linear multi-chromatic wave packet, interact to produce frequency sum and frequency difference components at the next order in amplitude. The frequency difference components, in turn, are responsible for the return flow. In physical terms, the kinematic and dynamic free surface boundary conditions, requiring no flow through the free surface and a uniform pressure on this surface, prevent a net deposition of fluid at the leading edge of the packet due to Stokes drift. It can be shown (van den Bremer & Taylor (in preparation)) that for deep water and in the limit of a narrow-banded spectrum the return flow is equal to the potential flow field corresponding to a series of sources and sinks of mass placed at the still water level, respectively, upstream and downstream of the centre of the wave packet with their strength determined by the local variation of the mass flux associated with the horizontal Stokes drift. As a result, a spatial separation of the two effects takes place: Stokes drift, the strength of which varies rapidly (exponentially) with depth, is the dominant effect at the free surface, whereas the return flow, which displays a much more slow decay with depth, dominates at a distance below.

For internal gravity waves on the continuum of equidensity surfaces in a stably stratified fluid the story is markedly different. In contrast to periodic surface gravity waves, horizontally and vertically periodic internal gravity waves that exist on stably stratified fluids do not display Stokes drift. In fact, the linear horizontally and vertically periodic (or planar) wave solutions satisfy the non-linear governing equations exactly, with no requirement for higher order corrections or induced-mean flow arising, unlike the case of surface gravity waves. The individual trajectory of a Lagrangian particles forms a straight line inclined at an angle to the horizontal, and the trajectory is completed as the local equidensity surface is displaced sinusoidally in time. The magnitude of the angle of inclination of the straight trajectory is determined by the ration of the frequency of the wave and the buoyancy frequency associated with the stratification.

For internal gravity wave groups that are vertically compact but remain periodic in

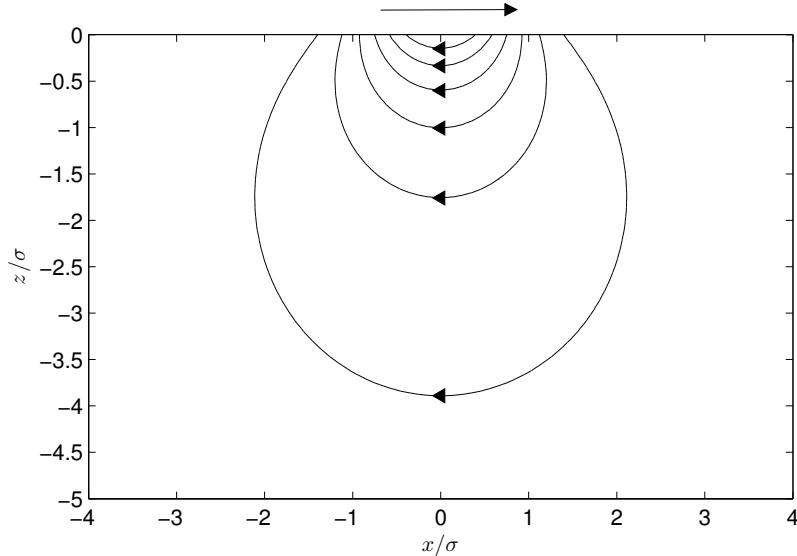


Figure 2: Streamlines corresponding to the (leftward) return flow below a (rightward) travelling surface gravity wave group from Longuet-Higgins & Stewart (1962) (see appendix A) for a Gaussian linear spectrum with bandwidth $\epsilon = (k_0\sigma)^{-1} = 1$ with σ denoting the bandwidth of the spectrum or the Gaussian half-width of the linear wave group and k_0 denoting the wave number of the peak of the spectrum (cf. figure 2 of McIntyre (1981)).

the horizontal, it is well established that a wave-induced mean flow (or a Stokes drift) exists and, in fact, plays a large role in determining the stability properties of such groups. Using Hamiltonian fluid mechanics, Scinocca & Shephard (1992) show that the horizontal wave-induced mean flow for such groups is given by:

$$\bar{u} = -\langle \xi \zeta \rangle, \quad (2)$$

where the angular brackets denote averaging over one horizontal wave length, $\xi(x, z, t)$ denotes the vertical displacement and $\zeta(x, z, t)$ denotes the vorticity. As for the Stokes drift for surface gravity waves (1), the induced mean flow in (2) is quadratic in the amplitude of the vertical displacement. Alternative means of deriving equivalent expressions to (2) include but are not limited to Stokes' circulation theorem (Sutherland (2010), §3.4.5), energy conservation relation (Bretherton (1969)) or from momentum flux divergence (see §3.1). Using the polarization relationships for linear internal gravity waves it can be shown that the induced mean flow \bar{u} in (2) is positive in the direction of the horizontal component of the group velocity.

Weakly nonlinear theory and fully nonlinear simulations have demonstrated that this wave-induced mean flow transiently Doppler-shifts the wave packet as it evolves, changing its structure and significantly altering where the wave packet breaks through modulational instability of the waves resulting from interaction with the self-induced mean flow. As for surface gravity waves (see for example Dysthe (1979)), the interaction with the horizontal

induced mean flow of horizontally periodic but vertically compact wave groups is well-captured by non-linear Schrödinger type evolution equations subject to an instability of the Benjamin-Feir type (Benjamin & Feir (1967)). For Boussinesq, non-Boussinesq and anelastic horizontally periodic wave packets Sutherland (2006) and Dosser & Sutherland (2011a, 2011b) have derived such equations and compared their predictive ability to fully non-linear simulations.

The framework of Lagrangian-mean flow, which underlies some of the terminology used in this report, was not formalized until Andrews & McIntyre (1978), who provided a generalized theory for the back effect of oscillatory disturbances upon the mean state termed the ‘generalized Lagrangian-mean’ (GLM). As noted by these authors, the concept of ‘Lagrangian mean’ is often required in a more general sense than its classical sense of the mean following a single fluid particle. What is of interest is the Lagrangian-mean flow described in terms of equations in Eulerian form, i.e. potentially as a function of position (x,z) and time t . When we refer to the Stokes drift velocity (cf. equation (2), which not a function of x only due to the periodic nature in that direction, but is a function of z) or the mass flux at a certain position x and time t , it is in fact the ‘generalized Lagrangian-mean’ of Andrews & McIntyre (1978), a hybrid Eulerian-Lagrangian description of wave mean-flow interaction, we implicitly adopt.

Internal gravity waves are widespread in both the ocean, in which the variation of salinity and temperature are responsible for the gradient in density with depth, and the atmosphere, which displays very significant temperature and density variation with height, where they may be generated by wind flow over mountain ranges or storms in the troposphere. Stokes drift associated with internal gravity wave packet may affect momentum transport and deposition indirectly through changing the criteria for breaking and resulting deposition of momentum to the environment, inclusion of which has improved predictions of mean zonal winds and temperatures in the middle atmosphere (McLandress (1998)). Its effects may also be direct through dispersion of tracers in the ocean ((Sanderson & Okubo (1988) and Holmes-Cerfon, Bühler & Ferrari (2011)).

This report considers the Stokes drift for a vertically and horizontally confined Boussinesq internal gravity wave packet in a linearly and stably stratified two-dimensional fluid of infinite vertical and horizontal extent and examines the existence and nature of a predicted return flow. It asks the question whether the results from horizontally periodic wave packets can be readily extended to the vertically and horizontally compact case or whether the wave-induced mean flow is accompanied by a return flow in a similar way to the surface gravity wave packet. For the benefit of simplicity, we apply the Boussinesq approximation, an assumption that is omnipresent in the literature on buoyancy driven flow and that assumes density variations are small and can be ignored except where they are responsible for the existence of the buoyancy force itself. The Boussinesq approximation is typically adequate for internal gravity waves in the ocean, in which the density may only vary by a small amount across the ocean’s depth, but inadequate in the atmosphere, in which individual waves will experience a significant decrease in background density as they rise and non-Boussinesq effects become important.

This report is laid out as follows. First §2 introduces the governing equations and discussed the assumptions made in their derivation. First, §3.1 reviews established results for wave packets that are compact in the vertical direction, but periodic in the horizontal direction and shows that for such packets there is an unbalanced induced mean flow in the horizontal direction. §3.2 extends this analysis based on momentum flux divergence to wave packets that are both vertically and horizontally compact. Using a separation of scales argument, we show that at leading order in bandwidth of the spectrum Stokes drift can only induce a global response in the horizontal direction and there can be no flow in the vertical direction, even if this flow is balanced by a return flow. In §4 we complete the discussion by deriving expressions for the local response in the vain of Longuet-Higgins & Stewart (1962), which would arrive at higher order in bandwidth in the perturbation expansion of §3.2. Finally, conclusions are drawn in §5.

2 Governing equations

We begin by making the usual Boussinesq approximation, that is, we ignore variation of density in the conservation equations except where it is responsible for the existence of the buoyancy force itself (cf. Spiegel & Veronis 1960). In a two-dimensional coordinate system (x, z) , where x denotes the horizontal coordinate and z denotes the vertical coordinate, the momentum equations in the x and z directions then become, respectively:

$$\frac{Du}{Dt} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \quad (3)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - g \frac{\Delta \rho}{\rho_0}, \quad (4)$$

where u and w denote the horizontal and vertical velocity, respectively, p the pressure, and gravity g acts in the negative z direction. The total density of the fluid ρ is the sum of background stratification density $\bar{\rho}$ and small variations from the background $\Delta \rho$, so that $\rho = \rho_0 + \Delta \rho$. Mass is conserved:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}), \quad (5)$$

where $\mathbf{u} = (u, w)$ and the fluid is assumed to be incompressible:

$$\frac{D\rho}{Dt} = 0. \quad (6)$$

Incompressibility (6) can be viewed as an additional assumption or as an implication of conservation of mass under the Boussinesq approximation. It is common to combine the (5) and (6) to give:

$$\nabla \cdot \mathbf{u} = 0. \quad (7)$$

We further assume the ambient is linearly stratified with (constant) buoyancy frequency N_0 :

$$N_0^2 = -\frac{g}{\rho_0} \frac{d\rho_0}{dz}. \quad (8)$$

Equation (6) can then be rewritten to give an equation for the density perturbation $\Delta\rho$:

$$\frac{D\Delta\rho}{Dt} = -w \frac{d\rho_0}{dz}. \quad (9)$$

It is convenient to rewrite density perturbations $\Delta\rho$ into a corresponding vertical displacement field:

$$\xi = -\frac{\Delta\rho}{\rho'_0(z)}, \quad (10)$$

so that (9) can be rewritten as:

$$\frac{D\xi}{Dt} = \frac{\partial\psi}{\partial x}, \quad (11)$$

where $\psi(x, z, t)$ is a stream function defined so that $u = -\partial\psi/\partial z$ and $w = \partial\psi/\partial x$. It is convenient to combine the horizontal and vertical momentum equations (3) and (4) into one equation in terms of two variables: the stream function ψ and the vertical displacement ξ . By taking the curl of momentum equation in the x and z direction, we obtain the equation for (baroclinic) generation of vorticity:

$$\frac{D\nabla^2\psi}{Dt} = -N_0^2 \frac{\partial\xi}{\partial x}, \quad (12)$$

where the right hand side denotes the total derivative of vorticity $\zeta = -\nabla^2\psi$. Finally, (12) and (11) can be combined into one:

$$\underbrace{\left[\frac{\partial^2}{\partial t^2} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right] + N_0^2 \frac{\partial^2}{\partial x^2} \right]}_{\equiv L} \psi = \nabla \cdot \underbrace{\left[\frac{\partial}{\partial t} [\zeta \nabla \times \psi] + N_0^2 \frac{\partial}{\partial x} [\xi \nabla \times \psi] \right]}_{\equiv \mathbf{F}}, \quad (13)$$

where the cross product of the stream function is the velocity vector in the (x, z) direction $\mathbf{u} = \nabla \times \psi$. We have thus defined the linear wave operator L and the non-linear vector \mathbf{F} . Together (11) and (13) can be used to capture the entire dynamics of the problem without further approximation compared to the governing equation for momentum (3), (4) mass (5) and incompressibility (6).

3 Global wave-induced mean flow

We begin (§3.1) by deriving an expression for the wave-induced mean flow for a wave group that is compact in the vertical direction but periodic in the horizontal direction, as illustrated in figure 3a, that is consistent with expression (2) derived by Scinocca & Shephard (1992) (see Sutherland (2010)). We extend this result to a group that is both compact in the vertical and the horizontal direction, as illustrated in figure 3b giving rise to what we show is a misleading result (§3.2.1). We then take a step back in §3.2.2 and pursue a separation of scales perturbation expansion in two variables: the amplitude of the displacement field and the bandwidth of the spectrum. Throughout we assume a Gaussian amplitude spectrum to produce the figures.

3.1 Vertically compact and horizontally periodic wave packets (review)

For internal gravity wave packets that are vertically compact but horizontally periodic, we consider the following vertical displacement:

$$\xi = \text{Re}[A(Z, T)e^{i(k_x + k_z z - \omega t)}], \quad (14)$$

where $A(Z, T)$ is the (complex) envelope of the packet (in the z -direction) with units of vertical displacement that evolves on the slow scale $Z = \epsilon_z(z - c_{g,z}t)$, where $c_{g,z}$ is the vertical group velocity and $\epsilon_z = (k_z \sigma_z)^{-1}$ is the (small) bandwidth parameter with k_z denoting the peak of the quasi-monochromatic spectrum and σ_z its bandwidth. The effects of dispersion are assumed to come in at higher order via the slow time scale $T = \epsilon_z^2 t$. From (11), we obtain expressions for the horizontal and vertical velocity at the same order:

$$u = \text{Re}[iA(Z, T)N_0 \sin(\theta_0)e^{i(k_x + k_z z - \omega t)}], \quad (15)$$

$$w = -\text{Re}[iA(Z, T)N_0 \cos(\theta_0)e^{i(k_x + k_z z - \omega t)}], \quad (16)$$

where we use the angle θ_0 to denote the angle between the horizontal and vertical components of the wave number vector of the carrier wave $\tan(\theta_0) = k_z/k_x$. Invoking the incompressibility assumption, the horizontal momentum conservation equation (3) can be written in flux form:

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \quad (17)$$

Because the packet has remained periodic in the x -direction, we average over the fast variation in that direction to obtain from (17):

$$\frac{\partial}{\partial t} \langle u \rangle + \frac{\partial}{\partial z} \langle uw \rangle = 0, \quad (18)$$

where the angular brackets denote averaging in the x -direction. Continuing the informal separation of scales argument, we replace the temporal derivative on the left-hand side by a derivative with respect to the slow vertical scale Z , $\partial/\partial t = \epsilon_z^2 \partial_T - \epsilon_z c_{g,z} \partial_Z$, where we ignore the ϵ_z^2 term. Due to the averaging over the derivative $\partial/\partial z$ has become a slow derivative $\partial/\partial Z$ as well and we have to leading order in ϵ :

$$\langle u \rangle = \frac{1}{c_{g,z}} \langle uw \rangle. \quad (19)$$

Substituting in for the horizontal and vertical velocities from (15) and (16), we obtain:

$$\langle u \rangle = \frac{k_x N_0}{2 \cos(\theta_0)} A^2(Z, T). \quad (20)$$

Using the polarization relationship for linear waves (in amplitude) it can be shown that (20) is equivalent to (2). We thus have a induced mean flow that arises because of the vertical divergence of the flux of horizontal momentum.

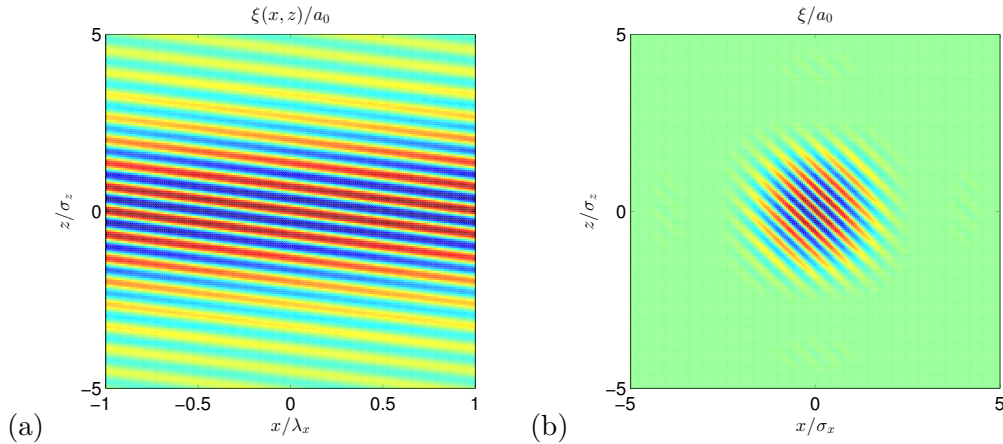


Figure 3: Linear vertical displacement ξ in (x, z) -space for (a) vertically compact but horizontally periodic Gaussian internal wave packet with $\epsilon_z = (\sigma_z k_{z,0})^{-1} = 0.1$ and (b) vertically and horizontally compact Gaussian packet with $\epsilon_x = (\sigma_x k_{x,0})^{-1} = \epsilon_z = (\sigma_z k_{z,0})^{-1} = 0.1$. In both cases $\omega_0/N_0 = 1/\sqrt{2}$ so that the group velocity vector points in the positive x and negative z direction at an angle of 45° .

3.2 Vertically and horizontally compact wave packets

3.2.1 Momentum flux divergence: a misleading picture

By introducing an additional slow scale $X = \epsilon_x(x - c_{g,x}t)$ and setting $\epsilon = \epsilon_x = \epsilon_z$, we extend the analysis in §3.1 to vertically and horizontally compact groups. The vertical displacement is given by:

$$\xi = \text{Re}[A(Z, X, T)e^{i(k_x x + k_z z - \omega t)}], \quad (21)$$

and the horizontal and vertical velocities can be found accordingly. Following a similar procedure to the one in §3.1, but now averaging over the fast time scale instead of the fast x scale and considering both the horizontal (3) and the vertical (4) momentum equation (in flux form), we obtain:

$$(u_{SD}, w_{SD}) = \frac{1}{2} |\mathbf{k}| |A|^2 \left(1, \frac{c_{g,z}}{c_{g,x}} \right). \quad (22)$$

The velocity field (22) is illustrated in figure4a and shows a field that is local to the Gaussian wave packet (unsurprisingly). As is evident from (22) its direction is that of the group velocity vector. Computing the (negative) divergence of this second-order (in amplitude) flow field (22) shows that it transports mass and deposits it at the leading edge of the wave packet. Evidently, the flow field cannot represent the entire picture: it is unbalanced in terms of mass and in terms of energy, as particles are transported against the stratification.

3.2.2 Separation of scales expansion in amplitude and bandwidth

We take a step back and consider a new separation of scales argument with the fast spatial and temporal scales denoted by x, z and t and the slow scales denoted by $X = \epsilon x, Z = \epsilon z$

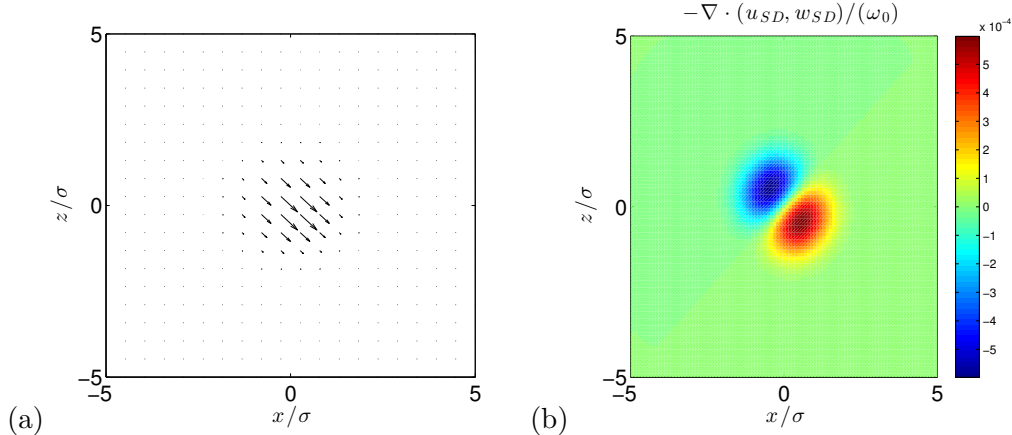


Figure 4: Unbalanced Stokes drift of a vertically and horizontally compact Gaussian packet with $\epsilon_x = (\sigma_x k_{x,0})^{-1} = \epsilon_z = (\sigma_z k_{z,0})^{-1} = 0.1$ showing (a) the flow field corresponding to (22) and (b) the negative divergence of this flow field illustrating there is a net transport of fluid in the direction of the group velocity. $\omega_0/N_0 = 1/\sqrt{2}$ so that the group velocity vector points in the positive x and negative z direction at an angle of 45° . The magnitude of the largest vector in (a) is $|\mathbf{u}_{SD}| = 1.2 \cdot 10^{-1} a_0 \omega_0$.

and $T = \epsilon t$. For tractability, we assume one universal bandwidth parameter $\epsilon = (k_x \sigma_x)^{-1} = (k_z \sigma_z)^{-1}$ with σ_x and σ_z denoting the horizontal and vertical bandwidth. We keep track of both the order in the bandwidth parameter ϵ , which we denote by a subscript, and the order in the amplitude of the signal, which we denote by a superscript. For example, $\xi_{(1)}^{(2)}$ denotes the component of the vertical displacement that is first order in bandwidth ϵ and second order in amplitude A . We do not make the assumption that $\epsilon = \alpha$, where $\alpha = kA$ with k denoting the magnitude of the wave number vector, an assumption that is commonly made to derive non-linear Schrödinger type of equations. We restrict our attention to first and second-order in amplitude, which we consider in turn.

Linear in amplitude $O(A)$

Our starting point is to assume the following form for the component of the vertical displacement that is first-order in amplitude:

$$\xi_{(0)}^{(1)} = \xi_{(0)}^{(1)} = \text{Re}[A(X, Z, T)e^{i(k_x x + k_z z - \omega t)}] \quad (23)$$

where the amplitude of the envelope A may be complex and we have set $\xi_{(n)}^{(1)} = 0$ for $n \geq 1$. From (11) we readily identify the counterpart of (23) that does not take any of the slow variation into account:

$$\psi_{(0)}^{(1)} = -\text{Re}\left[\frac{\omega}{k_x} A(X, Z, T)e^{i(k_x x + k_z z - \omega t)}\right]. \quad (24)$$

In order to satisfy (11) at every order in bandwidth we need to consider the effect of the derivatives acting on the slowly varying envelope. We use the following notation: we use

$\partial/\partial x$ to denote the combined effect of slow and fast derivatives, but let the subscripts in ∂_x and ∂_X denote only fast or slow derivatives, respectively. At first order in ϵ (11) becomes:

$$\epsilon \partial_T \xi_{(0)}^{(1)} = \epsilon \partial_X \psi_{(0)}^{(1)} + \partial_x \psi_{(1)}^{(1)}, \quad (25)$$

where we have used our assumption that $\xi_{(n)}^{(1)} = 0$ for $n = 1$. From (25) we have:

$$\psi_{(1)}^{(1)} = -\text{Re} \left[\frac{i}{k_x} \epsilon \left(A_T + \frac{\omega}{k_x} A_X \right) e^{i(k_x x + k_z z - \omega t)} \right]. \quad (26)$$

Equation (25) can be generalized for n th order in ϵ for $n > 1$:

$$\epsilon \partial_X \psi_{(n-1)}^{(1)} + \partial_x \psi_{(n)}^{(1)} = 0 \quad \text{for } n > 1, \quad (27)$$

which can be solved iteratively in combination with (26) to give:

$$\psi_{(n)}^{(1)} = -\text{Re} \left[\frac{i^n}{k_x^n} \epsilon^n \left(A_{X(n-1)T} + \frac{\omega}{k_x} A_{X(n)} \right) e^{i(k_x x + k_z z - \omega t)} \right] \quad \text{for } n > 1. \quad (28)$$

Having satisfied (11) at first-order in A and every order in ϵ , we turn to (13) and obtain the linear dispersion relation at zeroth order:

$$\frac{\omega^2}{N_0^2} = \frac{k_x^2}{k_x^2 + k_z^2}. \quad (29)$$

the envelope travelling at the group velocity at first order in ϵ and the effects of linear (in amplitude) dispersion at the next orders.

Second-order in amplitude $O(A^2)$

At second-order in amplitude we have from (13):

$$L\psi^{(2)} = \nabla \cdot \left[\frac{\partial}{\partial t} [\zeta^{(1)} \nabla \times \psi^{(1)}] + N_0^2 \frac{\partial}{\partial x} [\xi^{(1)} \nabla \times \psi^{(1)}] \right] \quad (30)$$

where the linear operator L is defined in (13) and we have so far included all terms in ϵ . Using the expressions for ψ_n^1 and ξ_n^1 for $n \geq 0$ derived above and after considerable manipulation, it can be shown that the right hand side is equal to zero at zeroth and first order in ϵ and periodic at second order and hence unable to provide the forcing required for a mean flow. The first non-zero and non-periodic only arise at even higher order.

3.2.3 Resulting global response

Considering the order (in the bandwidth ϵ) of all the terms in (30) we have:

$$\underbrace{\left[\frac{\partial^2}{\partial t^2} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right] \right]}_{O(\epsilon^4)} \underbrace{\left[\psi^2(X, Z, T) \right]}_{O(\epsilon^2)} = \underbrace{\nabla \cdot \left[\frac{\partial}{\partial t} [\zeta^{(1)} \nabla \times \psi^{(1)}] + N_0^2 \frac{\partial}{\partial x} [\xi^{(1)} \nabla \times \psi^{(1)}] \right]}_{O(\epsilon^4)}. \quad (31)$$

We thus have to leading order (second):

$$\partial_{XX}\psi^2 = 0, \rightarrow \psi^2 = f(Z)X + g(X), \quad (32)$$

where $f(Z)$ and $g(X)$ are arbitrary function of the slow scales X and Z . In combination with incompressibility (7), it is then easy to show that the only possible flow field at second order in ϵ , the order at which the Stokes drift in §3.2.1 is derived, that is not periodic and does not grow indefinitely in x or z , is the following:

$$u_2^2 = g(z), \quad w_2^2 = 0. \quad (33)$$

In words, at leading order in bandwidth the total (non-periodic) horizontal velocity that is the sum of the Stokes drift derived in §3.2.1 and a “return flow” $u^2 = u_{SD}^2 + u_{RF}^2$ can only be a function of the vertical coordinate z . It cannot display any variation in the horizontal direction. The total vertical velocity that is the sum of the Stokes drift velocity derived in §3.2.1 and “return flow” is zero $w^2 = w_{SD}^2 + w_{RF}^2 = 0$. The return flow thus exactly cancels out the vertical component of the Stokes drift at all points in space (x, z) but leaves long disturbances in the x -direction:

$$(u_{RF}, w_{RF}) = \left(-u_{SD} + \frac{1}{L_x} \int_0^{L_x} u_{SD} dx, -w_{SD} \right) \quad (34)$$

where L_x is the length of the domain in the x -direction. The Stokes drift is spread out horizontally over the domain and decreases with the size of the domain considered. In the limit of an infinite (computational) domain $L_x \rightarrow \infty$ the induced mean flow goes to zero, but may be finite in magnitude even in infinite domains provided the number of wave groups per unit length is finite. In reality, the horizontal extent of the disturbance is limited by the time scale for vertical propagation and the horizontal group velocity for long waves, effects both not considered herein.

4 Local circulation for broadbanded packets

4.1 Lagrangian effects

The presence of a second local and potentially much smaller effect becomes apparent from numerically integrating the equations of motion of a Lagrangian particle with respect to time:

$$\frac{d\Delta x}{dt} = u(x, z, t), \quad \frac{d\Delta z}{dt} = w(x, z, t), \quad (35)$$

where we obtain the horizontal and vertical velocities from linear theory:

$$u = \text{Re} [iA(Z, X, T)N_0 \sin(\theta_0) e^{i(k_x + k_z z - \omega t)}], \quad (36)$$

$$w = -\text{Re} [iA(Z, X, T)N_0 \cos(\theta_0) e^{i(k_x + k_z z - \omega t)}], \quad (37)$$

use a bivariate Gaussian distribution for $A(X, Z)$ and ignore the effects of dispersion via the additional slow scale T . Figure 5 shows the motion the Lagrangian particles undergo during the passing of the wave group. What is also evident from the figure is that the

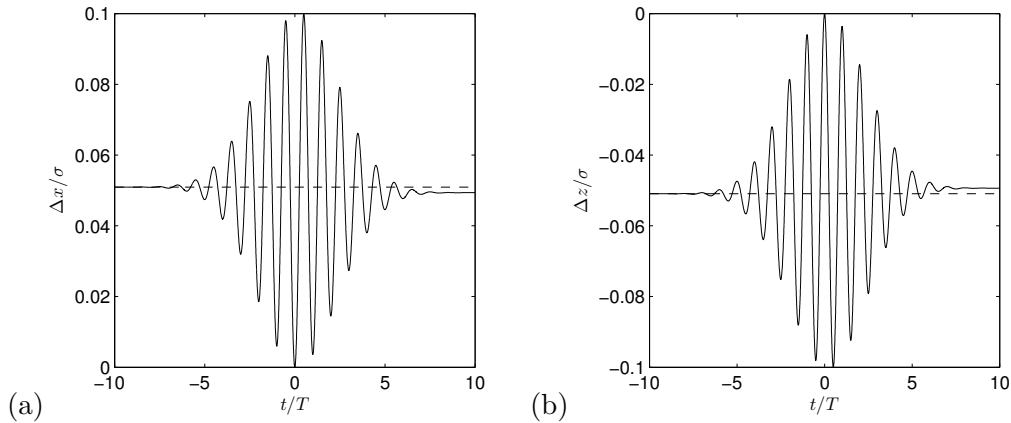


Figure 5: Displacement of Lagrangian particles in the (a) horizontal and (b) vertical direction due to the passing of an internal wave group obtained from time integration of (35) with linear (in amplitude) vertical and horizontal velocities (36-37). The Lagrangian particles are located at the focus point $x = 0$, $z = 0$ of the Gaussian wave group at the time of focus $t = 0$.

particles undergo a net displacement. Figure 6 shows the actual particle trajectory, which still forms a straight line in (x, z) -space, but a net displacement in the direction opposite to the group velocity vector. We set out to find the accompanying return flow, which must occur at higher order in bandwidth than the global effects discussed in §3.

4.2 Derivation of local circulation

A third order in ϵ and above the separation of scales argument laid out in §3.2 becomes very cumbersome involving a very large number of different terms. It becomes advantageous to pursue the expansion in Fourier space. As an analogue to the derivation for the return flow of surface gravity waves by Longuet-Higgins & Stewart (1962) reproduced in appendix A for completeness, we define the solution to the linearised equations for vertically and horizontally compact internal wave packet as the sum of individual terms:

$$\xi^{(1)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cos(k_{x,n}x + k_{z,m}z - \omega_{nm}t + \mu_{nm}), \quad (38)$$

where A is the two-dimensional matrix of amplitude coefficients. In contrast to the surface gravity wave case, where a single summation suffices, a double summation is required to represent the spatial structure of the linear internal wave packet. From the linearised version of (11), we obtain:

$$\psi^{(1)} = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_{nm}\omega_{nm}}{k_{x,n}} \cos(k_{x,n}x + k_{z,m}z - \omega_{nm}t + \mu_{nm}), \quad (39)$$

The inclusion of all the different spectral terms ensures the linearised (in amplitude) version of (11) is effectively satisfied at all order in the bandwidth parameter ϵ . In addition, all

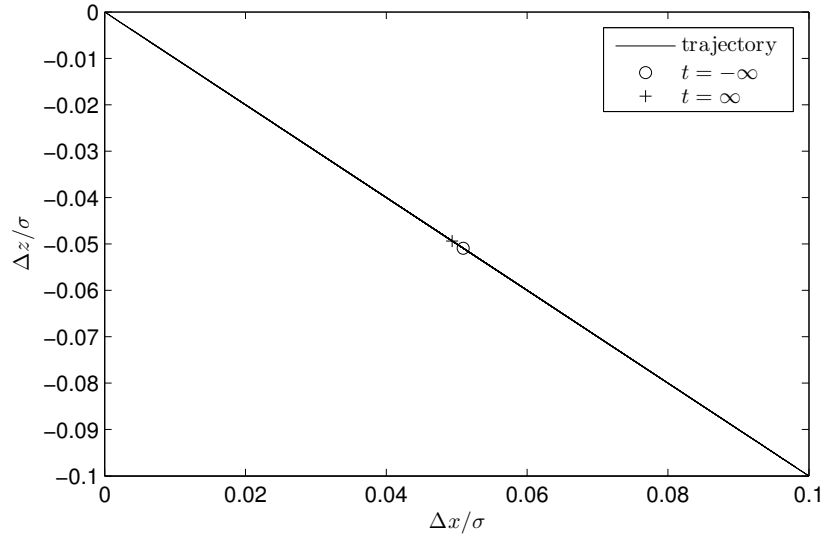


Figure 6: Trajectories of Lagrangian particles due to the passing of an internal wave group obtained from time integration of (35) with linear (in amplitude) vertical and horizontal velocities (36-37). The Lagrangian particles are located at the focus point $x = 0, z = 0$ of the Gaussian wave group at the time of focus $t = 0$. The initial particle, prior to the arrival of the wave group is denoted by an open circle and the final position is denoted by a + symbol. The net displacement is in the direction opposite to the group velocity vector.

the individual components satisfy the linear dispersion equation (29). From the linearised version of (13):

$$\frac{\omega_{nm}^2}{N_0^2} = \frac{k_{x,n}^2}{k_{x,n}^2 + k_{z,n}^2} \quad \text{for } \forall n, m \quad (40)$$

The full solution up to second-order in amplitude is given by:

$$\xi = \xi^{(1)} + \xi^{(2)} + O(A^3), \quad \psi = \psi^{(1)} + \psi^{(2)} + O(A^3), \quad (41a,b)$$

We set out to find $\xi^{(2)}$ and $\psi^{(2)}$ corresponding to $\xi^{(1)}$ (38) and $\psi^{(1)}$ (39). From (13) we have for $\phi^{(2)}$:

$$L\psi^{(2)} = \nabla \cdot \underbrace{\left[\frac{\partial}{\partial t} [\zeta^{(1)} \nabla \times \psi^{(1)}] + N_0^2 \frac{\partial}{\partial x} [\xi^{(1)} \nabla \times \psi^{(1)}] \right]}_{\equiv \mathbf{F}^{(2)}}, \quad (42)$$

where the linear operator L is defined in (13) and all the individual terms on the right hand side are linear in amplitude so that the right hand side itself becomes second-order in amplitude (cf. $\mathbf{F}^{(2)}$). After considerable manipulation it can be shown that the right hand side of (42) takes the form:

$$L\psi^{(2)} = \frac{N_0^2}{2} \sum_{n_1=1}^{\infty} \sum_{m_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_2=1}^{\infty} A_{n_1 m_1} A_{n_2 m_2} \Gamma_1 \Gamma_2 \sin(\Omega_{n_1 m_1} - \Omega_{n_2 m_2}), \quad (43)$$

where the phases are given by:

$$\Omega_{n_1 m_1} = k_{x,n_1} x + k_{z,m_1} z - \omega_{n_1 m_1} t + \mu_{n_1 m_1}, \quad \Omega_{n_2 m_2} = k_{x,n_2} x + k_{z,m_2} z - \omega_{n_2 m_2} t + \mu_{n_2 m_2}. \quad (44)$$

The coefficients Γ_1 and Γ_2 are given by:

$$\Gamma_1 = \omega_{n_1 m_1} k_{x,n_2} \left(\frac{\omega_{n_1 m_1}}{\omega_{n_2 m_2}} + \frac{k_{x,n_1}}{k_{x,n_2}} - 2 \right), \quad \Gamma_2 = k_{z,m_2} - k_{z,m_1} \frac{k_{x,n_2}}{k_{x,n_1}}. \quad (45)$$

Before we invert the linear operator in (43), we set out to simplify the right hand side of (43) by considering only its leading order variation in bandwidth, that is we only include the lowest order non-zero terms in $\Delta\omega = \omega_{n_1 m_1} - \omega_{n_2 m_2}$. The coefficients Γ_1 and Γ_2 (45) simplify to:

$$\Gamma_1 = \frac{2 \tan(\theta_0)}{c_{g,x}} \Delta\omega + O((\Delta\omega)^2), \quad \Gamma_2 = \frac{\omega_{n_1 m_1}}{c_{g,x}} (1 + \tan^2(\theta_0)) \Delta\omega + O((\Delta\omega)^2). \quad (46)$$

Inverting the linear operator in (43) requires multiplication of the right-hand side of (43) in Fourier space by:

$$L^{-1} \longleftrightarrow \frac{1}{(\omega_{n_2 m_2} - \omega_{n_1 m_1})^2 ((k_{x,n_2} - k_{x,n_1})^2 + (k_{z,m_2} - k_{z,m_1})^2) - N_0^2 (k_{x,n_2} - k_{x,n_1})^2}. \quad (47)$$

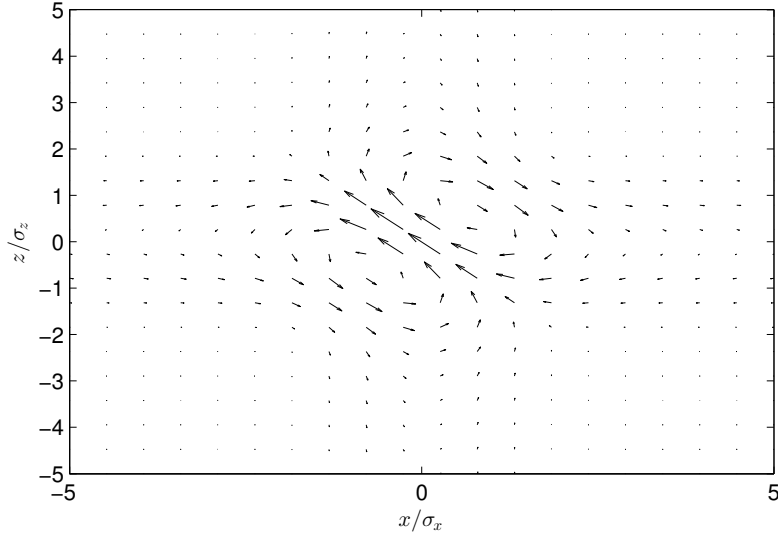


Figure 7: Local circulation flow field (u, w) corresponding to the stream function (48) for a Gaussian wave packet with bandwidth parameter $\epsilon_x = (\sigma_x k_{x,0})^{-1} = \epsilon_z = (\sigma_z k_{z,0})^{-1} = 0.1$ with $\sigma_x = \sigma_z$ denoting the bandwidth and $k_{x,0} = k_{z,0}$ denoting the peak of the spectrum. In particular, $\omega_0/N_0 = 1/\sqrt{2}$ so that the group velocity vector points in the positive x and negative z direction at an angle of 45° , exactly opposite to the largest arrows at the centre of the packet. The magnitude of the largest vector $|\mathbf{u}| = 5.6 \cdot 10^{-3} a_0 \omega_0$.

It is evident that the left-hand side term in the denominator is $(\Delta\omega)^4$, whereas the right-hand side is only $(\Delta\omega)^2$. Ignoring the former, the linear operator can be inverted to give the following leading-order stream function:

$$\psi^{(2)} = - \sum_{n_1=1}^{\infty} \sum_{m_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_2=1}^{\infty} A_{n_1 m_1} A_{n_2 m_2} \tan(\theta_0) (1 + \tan^2(\theta_0)) \omega_{n_1 m_1} \sin(\Omega_{n_1 m_1} - \Omega_{n_2 m_2}), \quad (48)$$

where the term includes information about both the velocity at which this flow field travels, namely the group velocity, and the local structure of the re-circulatory flow. Figure 7 shows the spatial structure of flow field (u, w) corresponding to the stream function (48). What is evident from this figure is a strongly localized flow through the centre of the wave packet and in the direction opposite to the group velocity vector, and a return flow in the opposite direction around the packet that is more spread out. Evidently, the flow field (48) is divergence free (volume is conserved), and leaves the energy of the system unchanged.

In our search for the leading order variation after (43), we have only excluded terms that describe the effect of linear (in amplitude) dispersion that could in principle be included but would modify the local circulation, a higher order phenomenon in bandwidth than the global induced mean flow of §3, slightly by introducing even higher order terms in bandwidth.

5 Conclusions

For vertically and horizontally compact internal wave groups in a linearly stratified ambient we have shown that two types of wave-induced mean flow can be distinguished: a global response consisting of horizontally long disturbances and no vertical motion and a local circulation that is associated with balanced horizontal and vertical motion. Both phenomena are second-order in amplitude, but the local circulation occurs at higher order in the bandwidth of the spectrum.

In the terminology of the surface gravity wave group, for which Stokes drift at the free surface is balanced by a return flow at depth giving rise to zero net depth-integrated momentum, the vertical component of the induced mean flow for internal wave packets is cancelled out locally by a return flow of equal magnitude and opposite sign with no motion in vertical as a result. In the horizontal direction the “return flow” acts to cancel out the horizontal structure of the Stokes drift, which is local to the packet, so that any variation of the horizontal induced mean flow along the horizontal direction disappears: the wave group induces long disturbances. As the domain under consideration is increased to infinity, the magnitude of these disturbances goes to zero. The local circulation, on the other hand, displays a behaviour that is reminiscent of that of surface gravity wave packets, and consists of a Stokes drift through the centre of the packet (in the direction opposite to the group velocity) that is balanced by a return flow around the packet.

A The return flow for surface gravity wave groups following Longuet-Higgins & Stewart (1962)

Longuet-Higgins & Stewart (1962) derive an expression for the return flow underneath a surface gravity wave group in finite water depth d without making any assumptions about the bandwidth of the underlying linear spectrum. We reproduce the derivation below assuming the water depth is large relative to all the linear components of the spectrum (the deep-water assumption).

A.1 Governing equations and boundary conditions

A two-dimensional body of water of infinite depth and indefinite lateral extent is assumed with a coordinate system (x, z) , where x denotes the horizontal coordinate and z the vertical coordinate measured from the undisturbed water level upwards. Inviscid, incompressible and irrotational flow is assumed and, as a result, the velocity vector can be defined as the gradient of the velocity potential $\mathbf{u} = \nabla\phi$, and the horizontal and vertical velocity as $u = \partial\phi/\partial x$ and $w = \partial\phi/\partial z$, respectively. The governing equation within the domain of the fluid is then Laplace:

$$\nabla^2\phi = 0 \quad \text{for } z \leq \eta(x, t), \quad (49)$$

where $\eta(x, t)$ denotes the free surface. The no-flow bottom boundary condition is:

$$\lim_{z \rightarrow -\infty} \frac{\partial\phi}{\partial z} = 0. \quad (50)$$

The kinematic free surface boundary condition (KFSBC) defines the free surface as moving with the particles located at the free surface such that particles located at the free surface stay there, i.e. $D(z - \eta)/Dt = 0$ or:

$$w - \frac{\partial\eta}{\partial t} - u \frac{\partial\eta}{\partial x} = 0 \quad \text{at } z = \eta(x, t). \quad (51)$$

Finally, the dynamic free surface boundary condition (DFSBC), which states that pressure at the free surface is constant and zero, fully closes the problem:

$$g\eta + \frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2 + w^2) = 0 \quad \text{at } z = \eta(x, t), \quad (52)$$

where gravity g acts in the negative z direction.

A.2 The solution to second-order in amplitude

A multi-chromatic solution to the governing equation (49) that satisfies the bottom boundary condition (50) exactly and satisfies the free surface boundary conditions (51) and (52) at first-order in amplitude is given by:

$$\xi^{(1)}(x, z, t) = \sum_{n=1}^{\infty} a_n \cos(k_n x - \omega_n t + \mu_n), \quad (53)$$

$$\xi^{(1)}(x, z, t) = \sum_{n=1}^{\infty} \frac{a_n \omega_n}{k_n} \sin(k_n x - \omega_n t + \mu_n), \quad (54)$$

where a_n , k_n , ω_n , μ_n denote the amplitude, wave number, wave frequency and phase of the individual components. All the individual components satisfy the linear dispersion equation:

$$\omega_n^2 = gk_n \quad \text{for } \forall n. \quad (55)$$

Retaining only terms that are second order in amplitude, the kinematic boundary condition (51) can be rewritten as:

$$\frac{\partial \phi^{(2)}}{\partial z} - \frac{\partial \eta^{(2)}}{\partial z} = \frac{\partial}{\partial x} \left(\eta^{(1)} \frac{\partial \phi^{(1)}}{\partial x} \right) \quad \text{at } z = 0, \quad (56)$$

where we have used $\partial^2 \phi^{(1)} / \partial x^2 = -\partial^2 \phi^{(1)} / \partial z^2$ from (49). The second-order solution still has to satisfy the linearised dynamic boundary condition (52):

$$\frac{\partial^{(2)}}{\partial t} + g\eta^{(2)} = 0. \quad \text{at } z = 0, \quad (57)$$

Combining (56) and (57), gives:

$$\frac{\partial \phi^{(2)}}{\partial z} + g \frac{\partial^2 \phi^{(2)}}{\partial t^2} = \frac{\partial}{\partial x} \left(\eta^{(1)} \frac{\partial \phi^{(1)}}{\partial x} \right) \quad \text{at } z = 0. \quad (58)$$

Equation (58), as pointed out by Dysthe (1979), in combination with the bottom boundary condition (50) and the governing equation (49), fully specify the solution for $\phi^{(2)}$. This solution is:

$$\phi^{(2)} = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} a_n a_m \omega_m e^{(k_m - k_n)z} \sin((k_n - k_m)x - (\omega_n - \omega_m)t + (\mu_n - \mu_m)), \quad (59)$$

where the requirement that $m > n$ ensures that the bottom boundary condition is met. Equation (59) can also be rewritten in terms of the corresponding stream function:

$$\psi^{(2)} = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} a_n a_m \omega_m e^{(k_m - k_n)z} \cos((k_n - k_m)x - (\omega_n - \omega_m)t + (\mu_n - \mu_m)), \quad (60)$$

Contours of constant values of the stream function (60) in (x, z) -space are shown in figure 2.

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