Stability of Shear Flow

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A look at energy stability, valid for all amplitudes, and linear stability for shear flows.

1 Nonlinear stability

Associated Navier-Stokes equation:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla P = \mathbf{F} + \nu \nabla^2 \mathbf{v} \quad \text{with} \quad \nabla \cdot \mathbf{v} = 0 \tag{1}$$

In this equation $\nu = R^{-1}$ is the nondimensional viscosity coefficient, where R is the Reynolds number. Let us assume a base flow $\mathbf{U}(\mathbf{x}, t)$ that is a known solution to equation (1) driven by the body force $\mathbf{F}$ (*e.g.* an imposed pressure gradient $\mathbf{F} = -\hat{\mathbf{x}}\, dP_0/dx$ in channel flow, or gravity for flow down an inclined channel) and/or the boundary conditions. Next we perturb the flow as $\mathbf{v} = \mathbf{U} + \mathbf{u}$ where $\mathbf{u} = (u, v, w)$ represents the perturbation. We plug this $\mathbf{v}$ into equation (1) which yields:

$$\partial_t \big(\mathbf{U} + \mathbf{u}\big) + \mathbf{U} \cdot \nabla \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \big(P + p\big) = \mathbf{F} + \nu \nabla^2 \big(\mathbf{U} + \mathbf{u}\big) \tag{2}$$

Since $\mathbf{U}$ is a solution of equation (1) the associated terms cancel and we get the perturbation equation:

$$\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u} \tag{3}$$

with the incompressible constraint $\nabla \cdot \mathbf{u} = 0$. For the domain V with fixed boundary ∂V, the boundary condition for $\mathbf{u}$ is homogeneous, namely, $\mathbf{u}\big|_{\partial V} = 0$ or periodic. Note that the decomposition $\mathbf{v} = \mathbf{U} + \mathbf{u}$ into a *base flow* plus a *perturbation* is different from the Reynolds decomposition $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$ into a *mean* plus a *fluctuation*. The base flow $\mathbf{U}$ is a solution of the Navier-Stokes equations and is independent of the perturbation $\mathbf{u}$, but the mean flow is $\bar{\mathbf{v}}$ is not a solution of Navier-Stokes and is coupled to the fluctuations $\mathbf{v}'$ through the Reynolds stresses.

In order to calculate the total kinetic energy of the perturbation, we multiply equation (3) by $\mathbf{u}$ and integrate over the domain V

$$\int_V \mathbf{u} \cdot \Big(\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u}\Big) dV = 0 \tag{4}$$

Direct computation using integration by parts and the incompressibility condition ($\nabla \cdot \mathbf{U} = \mathbf{0} \to \nabla \cdot \mathbf{u} = \mathbf{0}$) yields

$$\begin{aligned} \frac{d}{dt}\int_V \frac{|\mathbf{u}|^2}{2} dV &= \int_V -\mathbf{u}\cdot\nabla\mathbf{U}\cdot\mathbf{u}\, dV - \nu\int_V \nabla\mathbf{u} : \nabla\mathbf{u}^T\, dV \\ &\triangleq \underbrace{\int_V -\mathbf{u}\cdot\mathbf{S}\cdot\mathbf{u}\, dV}_{\text{Production}} - \underbrace{\nu\int_V |\nabla\mathbf{u}|^2\, dV}_{\text{Dissipation}} \end{aligned} \tag{5}$$

where $\mathbf{S}$ is the symmetric tensor strain rate tensor defined as $S_{ij} = \frac{1}{2}\big(\partial_i U_j + \partial_j U_i\big)$ and $u_i S_{ij} u_j = u_i(\partial_i U_j)u_j$ using Einstein summation and $\nabla\mathbf{u} : \nabla\mathbf{u}^T \triangleq (\partial_i u_j)(\partial_i u_j) = |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \triangleq |\nabla\mathbf{u}|^2$. Since the dissipation term is always positive, if the production term is negative or zero the the flow is *absolutely stable*, that is, stable to any perturbation $\mathbf{u}$.

Example: Rigid body rotation is absolutely stable, since the production term is 0. In this case

$$\begin{aligned} &\mathbf{U} = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \nabla\mathbf{U} = \begin{pmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &S_{ij} = \frac{1}{2}\big(\partial_i U_j + \partial_j U_i\big) \Rightarrow \mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

Definition (growth rate): We can think of the right hand side of (5) normalized by $2E \triangleq \int_V |\mathbf{u}|^2 dV$ as a *growth rate* since if the perturbation had the form $\mathbf{u} \triangleq e^{\lambda t}\widehat{\mathbf{u}}(\mathbf{x})$, as would be the case for time independent $\mathbf{U}$ in the linear limit, we would have $|\mathbf{u}|^2 = \exp(2\sigma t)|\widehat{\mathbf{u}}|^2$ and $(2E)^{-1}dE/dt = \sigma = \Re(\lambda)$, so σ, the real part of λ, is called the *growth rate*. We have $-\mathbf{u}\cdot\mathbf{S}\cdot\mathbf{u} \leqslant \lambda_{\max}\mathbf{u}\cdot\mathbf{u}$ where $\lambda_{\max}$ is the largest eigenvalue of the real and symmetric $(-\mathbf{S})$. Manipulating the right hand side of equation (5) gives

$$\begin{aligned} \sigma &\triangleq \frac{\int_V -\mathbf{u}\cdot\mathbf{S}\cdot\mathbf{u} dV}{\int_V |\mathbf{u}|^2 dV} - \frac{\nu\int_V |\nabla\mathbf{u}|^2 dV}{\int_V |\mathbf{u}|^2 dV} \\ &\leqslant \frac{\lambda_{\max}\int_V |\mathbf{u}|^2 dV}{\int_V |\mathbf{u}|^2 dV} - \frac{\nu\int_V |\nabla\mathbf{u}|^2 dV}{\int_V |\mathbf{u}|^2 dV} \leqslant \lambda_{\max} \end{aligned} \tag{7}$$

and this provides a simple upper bound on the growth rate of any instability.

Theorem (Serrin 1959): For any steady solution $\mathbf{U}$ there exists a critical Reynolds number $Re_1 > 0$ such that for any flow with $Re \leqslant Re_1$, the system is absolutely stable. See [2, §53.1] or [1, §9.6].

Next let's turn to shear base flows, *i.e.* $\mathbf{U} = U(y)\hat{\mathbf{x}}$ and

$$\mathbf{S} = \begin{pmatrix} 0 & \frac{U'}{2} & 0 \\ \frac{U'}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad U' \triangleq \frac{dU}{dy} \tag{8}$$

The corresponding kinetic energy takes the form

$$\frac{d}{dt}\int_V \frac{|\mathbf{u}|^2}{2} dV = \int_V -uv\frac{dU}{dy}dV - \nu \int_V |\nabla \mathbf{u}|^2 dV \tag{9}$$

that is very similar to the fluctuation energy equation derived in lecture 2, but again the production term there involved the mean shear rate $d\overline{U}/dy$ that depends on the Reynolds stress $\overline{uv}$, while here we have the base shear rate dU/dy that is independent of uv. For shear flows, the growth rate $\sigma < \max(U'/2)$ (assuming $U' \geq 0$) and this maximum would require very large Reynolds numbers $\nu = 1/R \to 0$ and $u = -v$ with $w = 0$, localized near the max of U'. In the case of nondimensional Couette flow $U(y) = y$, the energy equation reads

$$\frac{d}{dt}\int_V \frac{|\mathbf{u}|^2}{2} dV = \int_V -uv\, dV - \nu \int_V |\nabla \mathbf{u}|^2 dV \tag{10}$$

From this equation it can be seen that $-uv > 0$ occurring somewhere in the domain V is a necessary condition for instability. Turning to the energy stability of shear flows, if we define the critical value ν_E

$$\nu_E \triangleq \max \frac{\displaystyle\int_V -uv\frac{dU}{dy}dV}{\displaystyle\nu \int_V |\nabla \mathbf{u}|^2 dV} \tag{11}$$

it directly follows that

$$\frac{d}{dt}\int_V \frac{|\mathbf{u}|^2}{2} dV \leqslant (\nu_E - \nu) \int_V |\nabla \mathbf{u}|^2 dV. \tag{12}$$

The inequality (12) shows that the perturbation is stable if $\nu_E < \nu \Leftrightarrow R < 1/\nu_E \triangleq R_E$. This is a sufficient condition for stability and is known as the *absolute stability threshold*. Therefore an argument for absolute stability turns into an optimization problem (11) with the constraints $\nabla \cdot \mathbf{u} = \mathbf{0}$ and $\mathbf{u}\big|_{\partial V} = 0$

Remark: For Couette flow, the critical Reynolds number for absolute stability is about 20.7, see [2, §53.1].

2 Linear stability

The flow is decomposed into a base flow $\mathbf{U}$ and a perturbation about the base flow $\mathbf{u}$

$$\mathbf{v} = \mathbf{U} + \mathbf{u}. \tag{13}$$

Plugging into the Navier-Stokes equations and neglecting the quadratic nonlinearity $\mathbf{u} \cdot \nabla \mathbf{u}$ gives

$$\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \nabla p = \nu \nabla^2 \mathbf{u}. \tag{14}$$

The base flow is now taken to be a shear flow, $\mathbf{U} = U(y)\hat{\mathbf{x}}$. Taking the curl of (14) and dotting with the vertical unit vector $\hat{\mathbf{y}}$ gives

$$(\partial_t + U\partial_x - \frac{1}{R}\nabla^2)\eta = -\partial_z v \, \frac{dU}{dy} \tag{15}$$

where $\eta = \hat{\mathbf{y}} \cdot \nabla \times \mathbf{u}$ is the vertical component of the vorticity and $v = \hat{\mathbf{y}} \cdot \mathbf{u}$ is the vertical component of the perturbation velocity. Taking the curl of (14) twice and dotting with the vertical unit vector gives

$$(\partial_t + U\partial_x - \frac{1}{R}\nabla^2)\nabla^2 v - \partial_x v \frac{d^2U}{dy^2} = 0. \tag{16}$$

Equations (15) and (16) are known as the Squire and Orr-Sommerfeld equations, respectively. Note that the v equation (16) is decoupled from the η equation (15). There are two basic kinds of boundary conditions at the walls of the channel. One is no-slip boundary condition $u = v = w = 0$ which implies there is no perturbation at the walls. In the Orr-Sommerfeld equation this boundary condition takes the form $v = 0$, $v_y = -u_x - w_z = 0$ and $\eta = u_z - w_x = 0$. The other is 'free slip' boundary conditions (*i.e.* stress or Neumann boundary conditions on the full flow) $v = u_y = w_y = 0$ which implies $v = 0$, $\eta_y = u_{yz} - w_{xz} = 0$ and $v_{yy} = -u_{xy} - w_{zy} = 0$ at the walls of the channel. Next we turn to the Fourier analysis of the Orr-Sommerfeld system, since $U = U(y)$ only, equations (15) and (16) admit solutions of the form

$$\eta(x, y, z, t) = \hat{\eta}(y)e^{\lambda t}e^{i(\alpha x + \gamma z)}$$
$$v(x, y, z, t) = \hat{v}(y)e^{\lambda t}e^{i(\alpha x + \gamma z)}$$

where λ is a complex-valued growth rate, α and γ are the real streamwise and spanwise wavenumbers, respectively, and $\hat{\eta}(y)$ and $\hat{v}(y)$ are complex functions. Plugging the above forms of v and η into equations (15) and (16) gives

$$\left[\lambda + i\alpha U - \frac{1}{R}(D^2 - k^2)\right]\hat{\eta} \quad = \quad -i\gamma\hat{v}U' \tag{17}$$

$$\left[\lambda + i\alpha U - \frac{1}{R}(D^2 - k^2)\right](D^2 - k^2)\hat{v} - U'' i\alpha\hat{v} \quad = \quad 0 \tag{18}$$

where a prime indicates a y-derivative, $D = d/dy$, and $k^2 = \alpha^2 + \gamma^2$. Equation (18) can be simplified by multiplying through by k/α

$$\left[\widetilde{\lambda} + ikU - \frac{1}{\widetilde{R}}(D^2 - k^2)\right](D^2 - k^2)\hat{v} - U'' ik\hat{v} = 0 \tag{19}$$

where $\widetilde{\lambda} = \lambda k/\alpha$ and $\widetilde{R} = R\alpha/k$.

Squire's theorem. Equation (19) is (18) with $\alpha \equiv k$ and rescaled growth rate and Reynolds number. Therefore a three dimensional perturbation with wavenumbers (α, γ) at Reynolds number R with growth rate λ is mathematically equivalent to a two dimensional perturbation with wavenumbers $(k, 0)$ but with Reynolds number $\widetilde{R} = R\,\alpha/k \leq R$ and growth rate

$\Re(\widetilde{\lambda}) = \Re(\lambda)k/\alpha \geq \Re(\lambda)$. In other words, for any 3D unstable mode (α, γ) there is a 2D unstable mode $(\sqrt{\alpha^2+\gamma^2}, 0)$ with *larger* growth rate at a *lower* Reynolds number. This is Squire's Theorem [3], [2]. Another way to derive this result, is to let $\alpha U(y) = k\widetilde{U}(y)$ in (19) and conclude that a 3D perturbation is equivalent to a 2D perturbation with a weaker shear flow $\widetilde{U}(y) = \alpha U(y)/k$.

Furthermore, it is easy to show that the homogeneous η equation, that is (17) with $v = 0$ has only damped modes (multiply the homogeneous equation by η^* the conjugate of η, integrate from wall to wall and add the complex conjugate of the result to show that $\lambda + \lambda^* = 2\sigma \leq 0$). This is physically obvious since (17) is an advection diffusion equation when $v = 0$, in fact the decay of $\int_V \eta^2 dV$ for $v = 0$ can be shown for the full linear PDE (15). Thus the eigenvalue problem for (15), (16), reduces to the consideration of the Orr-Sommerfeld equation (19) for 2D perturbations only. Note that equation (17) with $v \neq 0$ exhibits *transient growth* for 3D perturbations with $\partial_z v \neq 0$, as discussed in lecture 1. The physical mechanism behind this is the redistribution of streamwise velocity $U(y)$ by the perturbation v to create u perturbations and $\eta = \partial_z u - \partial_x w$.

3 Energy equation

From Squire's theorem it suffices to consider (18) with $k = \alpha$

$$\left[\lambda + i\alpha U - \frac{1}{R}(D^2 - \alpha^2)\right](D^2 - \alpha^2)\hat{v} - U'' i\alpha\hat{v} = 0. \tag{20}$$

The equation for the complex conjugate $\hat{v}^*$ reads

$$\left[\lambda^* - i\alpha U - \frac{1}{R}(D^2 - \alpha^2)\right](D^2 - \alpha^2)\hat{v}^* + U'' i\alpha\hat{v}^* = 0. \tag{21}$$

since $U(y)$, α and R are real and where λ^* is the complex conjugate of λ. Doing the following surgery: $\hat{v}^* \cdot (20) + \hat{v} \cdot (21)$ and integrating from the bottom of the domain $(y = y_1)$ to the top of the domain $(y = y_2)$ yields

$$(\lambda + \lambda^*)\underbrace{\int_{y_1}^{y_2} \left(|Dv|^2 + \alpha^2|v|^2\right) dy}_{\text{kinetic energy}} = \underbrace{\int_{y_1}^{y_2} U'T dy}_{\text{production}} - \underbrace{\frac{2}{R}\int_{y_1}^{y_2} |\phi|^2 dy}_{\text{dissipation}}. \tag{22}$$

Here $T \triangleq i\alpha(\hat{v}D\hat{v}^* - \hat{v}^*D\hat{v}) \equiv -\alpha^2\overline{uv}$, and $\phi \triangleq (D^2 - \alpha^2)\hat{v}$ with

$$\int_{y_1}^{y_2} |\phi|^2 dy = \int_{y_1}^{y_2} |D^2 v - \alpha^2 v|^2 dy = \int_{y_1}^{y_2} \left(|D^2\hat{v}|^2 + 2\alpha^2|D\hat{v}|^2 + \alpha^4|\hat{v}|^2\right) dy \geq 0. \tag{23}$$

It is noted that when doing the integration by parts, the boundary condition $\hat{v}\big|_{\partial V} = 0$ is used, and either $D\hat{v}\big|_{\partial V} = 0$ or $D^2\hat{v}\big|_{\partial V} = 0$ can be applied to lead the same equality (22). In other words, (22) holds for both *no-slip* and *free-slip* boundary conditions.

Equation (22) is simply the version of (9) for 2D eigensolutions $v = \hat{v}(y)e^{\lambda t}e^{i\alpha x}$ with $u = \hat{u}(y)e^{\lambda t}e^{i\alpha x} = (i/\alpha)D\hat{v}\ e^{\lambda t}e^{i\alpha x}$ to satisfy $\partial_x u + \partial_y v = 0$ and the 'Reynolds stress' for such a perturbation is

$$-\overline{uv} = -\left(\hat{u}^*\hat{v} + \hat{u}\hat{v}^*\right)e^{2\sigma t} = \frac{i}{\alpha}\left(\hat{v}D\hat{v}^* - \hat{v}^*D\hat{v}\right)e^{2\sigma t} \equiv \frac{T}{\alpha^2}e^{2\sigma t} \tag{24}$$

where $2\sigma = \lambda + \lambda^*$ and $\overline{uv}$ is the horizontal average of uv. Likewise, $\phi = (D^2 - \alpha^2)\hat{v}$ is effectively the z component of vorticity $\omega_z = \partial_x v - \partial_y u = \left(i\alpha\hat{v} - (i/\alpha)D^2\hat{v}\right)e^{\lambda t}e^{i\alpha x} = (-i/\alpha)\phi(y)\,e^{\lambda t}e^{i\alpha x}$.

For *free-slip* boundary conditions, $\hat{v} = D^2\hat{v} = 0$ at the walls, we can derive a useful form of the *enstrophy equation*, *i.e.* an equation for the integral of vorticity squared. Consider $\int_{y_1}^{y_2}\left[(D^2-\alpha^2)\hat{v}^*\cdot(20) + (D^2-\alpha^2)\hat{v}\cdot(21)\right]dy = 0$ to obtain

$$(\lambda + \lambda^*)\int_{y_1}^{y_2}|\phi|^2 dy = \underbrace{-\int_{y_1}^{y_2} U''' T\, dy}_{\text{production}} - \underbrace{\frac{2}{R}\int_{y_1}^{y_2}\left[|D\phi|^2 + \alpha^2|\phi|^2\right]dy}_{\text{dissipation}\,\geq 0} \tag{25}$$

It follows directly that $U''' = 0$ implies linear stability for free-slip, that is $2\sigma = \lambda + \lambda^* \leq 0$. In other words plane Couette flow $U(y) = y$ and plane Poiseuille flow $U(y) = 1 - y^2$, in $-1 \leq y \leq 1$, are linearly stable for free-slip (*i.e.* imposed stress) boundary conditions as well as any combination of Couette and Poiseuille $U(y) = a + by + cy^2$, among other flows. We'll discuss this further in the next lecture.

The enstrophy equation (25) only holds under free-slip boundary condition, since $\hat{v}\big|_{\partial V} = D^2\hat{v}\big|_{\partial V} = 0$ leads to the cancelation of the boundary terms in integration by parts but a boundary term of indefinite sign subsists for no-slip $\hat{v} = D\hat{v} = 0$ at the walls. The physical meaning of these boundary terms is that vorticity can be generated (or destroyed) at the walls for no-slip but not for free-slip.

References

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[2] P. Drazin and W. Reid, *Hydrodynamic Stability*, Cambridge University Press, Cambridge, UK, 1981.

[3] H. Squire, *On the stability of three-dimensional disturbances of viscous flow between parallel walls*, Proc. Roy. Soc. Lond. A, 142 (1933), pp. 621–628.