Maximal heat transport in Rayleigh–Bénard convection: reduced models, bifurcations, and polynomial optimization

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In the turbulent regime of Rayleigh–Bénard convection, steady 2D solutions (which are unstable) transport heat about as quickly as fully turbulent 3D convection, provided the horizontal period of the steady solutions is chosen to maximize heat transport [3]. This suggests the possibility that steady 2D states might actually transport more heat than any time-dependent flow. If this is true for solutions of the PDEs governing Rayleigh–Bénard convection, proving it would likely be very difficult. (There are methods for rigorously bounding heat transport [1] in the PDEs, but the bounds are somewhat conservative.) However, we can pose the analogous question in a sequence of ODE models of increasing dimension, which approximate the full PDEs increasingly well. Determining which states maximize heat transport in the ODE case is easier for two reasons. First, it is less expensive to compute various solution branches in the bifurcation structure. Second, methods of polynomial optimization can be used to produce computer-assisted upper bounds on heat transport that are very precise.

The first part of the project is to derive a systematic hierarchy of increasingly accurate ODE models of Rayleigh–Bénard convection for certain natural boundary conditions by truncating Fourier expansions of the PDE variables. The truncations we will use are not arbitrary: we want truncations leading to ODEs that preserve certain properties of the PDEs as in [4]. Then we will study some of our models in detail. The smallest model in the hierarchy will be the famous Lorenz equations, which are well studied already and provide guidance, so we will start by studying the second-smallest system of ODEs.

Each ODE model can be studied from two complementary angles, which can be pursued in parallel. One angle is to build an understanding of the various types of solutions to the ODE model, not only by numerical integration but by using numerical bifurcation analysis to find various steady and time-periodic solutions that may or may not be stable. This can be done relatively easily using bifurcation analysis software such as MatCont. The other angle is to use computational methods of polynomial optimization to produce global upper bounds on heat transport in the ODE model. This too can be done using existing software; it has has been done for the Lorenz equations already [2], and we can build on the code used in that study. For a given ODE model, if we can compute a precise upper bound on heat transport and also find a particular steady state whose heat transport is equal to the upper bound, then this proves that maximal heat transport is achieved by steady states, as opposed to any chaotic (i.e. "turbulent") or other time-dependent solution.

References

[1] C. R. Doering and P. Constantin. Variational bounds on energy dissipation in incompressible flows. III. Convection. *Phys. Rev. E*, 53:5957–5981, 1996.

[2] D. Goluskin. Bounding averages rigorously using semidefinite programming: mean moments of the Lorenz system. *J. Nonlinear Sci.*, 28:621–651, 2018.

[3] D. Sondak, L. M. Smith, and F. Waleffe. Optimal heat transport in Rayleigh–Bénard convection. *J. Fluid Mech.*, 784:565–595, 2015.

[4] J.-L. Thiffeault and W. Horton. Energy-conserving truncations for convection with shear flow. *Phys. Fluids*, 8:1715, 1996.