The dynamics of a large pendulum chain

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The single pendulum mass is a classical paradigm in elementary dynamical systems. The "double pendulum" is considerably more complicated. Two identical masses are enough to display chaotic dynamics; *i.e.*, non-periodic motion, with sensitive dependence on initial conditions. Three and four masses show somewhat similar behaviour as two masses. The motion is more complex, but not obviously different in a qualitative way.

Very interesting things begin to emerge with large numbers of masses. Around 100 masses, and definitely around 1000, the system starts to display "thermodynamic" or "continuum" behaviour. Even though the system is perfectly frictionless, the small-scale motions act as a sink of energy for the large scales, which can now damp away. And even though the chain is perfectly inextensible, the system can begin to display wave-like bouncing motions.

Moreover, the equations and dynamics for the chain closely resemble stratified hydrodynamics in many ways. The system variables comprise a set of local angles and velocities of each link, along with a tension in each segment.

$$\boldsymbol{\theta}(t) \equiv \begin{bmatrix} \theta_0(t) \\ \vdots \\ \theta_{N-1}(t) \end{bmatrix}, \quad \boldsymbol{\omega}(t) \equiv \begin{bmatrix} \omega_0(t) \\ \vdots \\ \omega_{N-1}(t) \end{bmatrix}, \quad \boldsymbol{T}(t) \equiv \begin{bmatrix} T_0(t) \\ \vdots \\ T_{N-1}(t) \end{bmatrix}. \tag{1}$$

The dynamical equations evolve the such that,

$$\frac{d\boldsymbol{\theta}}{dt} = \boldsymbol{\omega}, \qquad \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\mathcal{D}}(\boldsymbol{\theta}) \cdot \boldsymbol{T} - \sin(\theta_0)\, \boldsymbol{e}_0 \tag{2}$$

The tension is satisfies an instantaneous balance,

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) \cdot \boldsymbol{T} = \boldsymbol{\omega}^2 + \cos(\theta_0)\, \boldsymbol{e}_0 \tag{3}$$

Operators acting on the tension are very close to 1st and 2nd derivative operators that depend on the angles.

$$\boldsymbol{\mathcal{D}}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & \sin(\theta_0 - \theta_1) & \dots & 0 \\ -\sin(\theta_0 - \theta_1) & 0 & \ddots & \vdots \\ \vdots & \ddots & 0 & \sin(\theta_{N-2} - \theta_{N-1}) \\ 0 & \dots & -\sin(\theta_{N-2} - \theta_{N-1}) & 0 \end{bmatrix} \tag{4}$$

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & -\cos(\theta_0 - \theta_1) & \dots & 0 \\ -\cos(\theta_0 - \theta_1) & 2 & \ddots & \vdots \\ \vdots & \ddots & 2 & -\cos(\theta_{N-2} - \theta_{N-1}) \\ 0 & \dots & -\cos(\theta_{N-2} - \theta_{N-1}) & 2 \end{bmatrix}. \tag{5}$$

Efficient code already exists that can simulate these set of equations. The code could also be adapted easily to include many other effects.

The equations are like stratified hydrodynamics in a number of ways, with some interesting twists.

- In a stratified fluid, gravity pull everything down, and pressure *pushes* it up in hydrostatic balance.
- Gravity also pulls the chain down, but tension supports the system by *pulling* from above.
- Near the resting (downward hanging) state, the system allows linear waves. These are the analogue of gravity waves.
- If the system starts off with a finite amount of energy, the large-scales transfer energy to small-scales. The system *cascades* energy down-scale.
- In the frictionless model, down-scale energy transfer acts to thermalise the system, which gives rise to finite temperature.
- At finite temperature, the chain coils to an average length that is shorter than the total length.
- The masses are not located in their original order. The system is mixed.
- The average length can extend if additional force is applied. The system develops longitudinal waves, even though the original chain is inextensible.
- The emergent tension is the result of configurational entropy and finite temperature.

$$F \mathrm{d}L \;=\; \Theta\, \mathrm{d}S \tag{6}$$

There are a couple more speculative ideas this project can explore.

- If we include explicit dissipation and external forcing, then the system can reach different steady states.
- The hypothesis is that the kinetic energy spectrum should be close to a turbulent Kolmogorov spectrum.
- If true, the pendulum chain could be a good model for vortical turbulence in one dimension.
- It would be perhaps possible to push the resolution to 10,000 links.