# Fluid-Structure Interactions in the Living Environment

**Schedule**:

1. June 20: [MS] A primer on continuum and fluid mechanics. Mass conservation, momentum balance, the Eulerian and Lagrangian frame. Boundary conditions.
2. June 21: [MS] Canonical fluid-structure problems: elastic structures interacting with high-speed flows. Flags, streamlining. Mathematical approaches: boundary integral methods, unsteady Kutta condition, elasticity, conformal mapping methods.
3. June 22: [MS] Flapping flight: symmetry-breaking and the transition to flapping flight. Studying collective flight through simple models and experiments.
4. June 23: [PH] High Re fluid-structure interactions in sports: sailing, ski jumping, cycling, kite boarding.
5. June 24: [PH] Low Reynolds number swimming introduction: RFT and a slender-body theory teaser, three-link swimmer, single flagellum, two flagella, optimization.
6. June 27: [MS] Low Reynolds number phenomena. Nonlocal Slender-body theory and numerical methods for many body interactions; Buckling of elastic bodies by flow, and anomalous stresses.
7. June 28: [MS] Collective behavior at low Reynolds number. Bioactive suspensions, simulations, continuum theories. Fluid-structure interactions in cellular biomechanics.
8. June 29: [PH] Thin films with elastic boundaries: crawling, peeling, adhesion, soft objects moving near rigid boundaries.
9. June 30: [PH] Hydrodynamics of textured surfaces: hairy textures, symmetry-breaking, Darcy-Brinkman flow.
10. June 31: [PH] TBD

**Some useful reference texts**:

- Incompressible Fluid Dynamics:

  George Batchelor – *An Introduction to Fluid Dynamics*

  D.J. Acheson – *Elementary Fluid Dynamics*

  C. Pozrikidis – *Boundary Integral and Singularity Methods for Linearized Viscous Flow*

- Complex Fluids and Solids

  R.G. Larson – *The Structure and Rheology of Complex Fluids*

  G.A. Holzapfel – *Nonlinear Solid Mechanics*

  Doi & Edwards – *The Theory of Polymer Dynamics*

- Bio Mechanics, Fluids, Locomotion

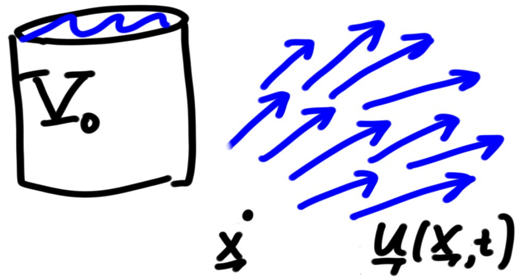
  S. Childress – *Mechanics of Swimming and Flying*

  S. Vogel – *Life in Moving Fluids*

  S. Vogel – *Comparative Biomechanics*

  R.M. Alexander – *Principles of Animal Locomotion*

# (1) Basic concepts of fluid and continuum mechanics




**The volume of fluid $V$ and the velocity field**.

Consider a volume $V$ filled with a fluid or continuous material. At each time $t$ and at each point $\mathbf{x}$ the fluid has a velocity $\mathbf{u}(\mathbf{x},t)$ and density $\rho(\mathbf{x},t)$. Describing the fluid flow as passing through a fixed (lab) coordinate frame is called the *Eulerian frame.*

## *Notation:*

$$\mathbf{x} = (x,y,z) = (x_1,x_2,x_3)$$

$$\mathbf{u} = (u,v,w) = (u_1,u_2,u_2)$$

## The basic constituents of the velocity field – translation, deformation, rotation

Consider a steady flow, fixing $\mathbf{x}$ and considering a nearby point $\mathbf{x}+\mathbf{r}$:

$$\mathbf{u}(\mathbf{x}+\mathbf{r}) = \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})\,\mathbf{r} + O(|\mathbf{r}|^2)$$

break up into symmetric and anti-symmetric parts

$$\approx \mathbf{u}(\mathbf{x}) + \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)\mathbf{r} + \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^\top)\mathbf{r}$$

$$\approx \mathbf{u}(\mathbf{x}) + \mathbf{E}\,\mathbf{r} + \mathbf{W}\,\mathbf{r}$$

w. $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$ called the rate-of-strain tensor, and $E_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ (the symmetric rate-of-strain tensor) and $W_{ij} = (\partial u_i/\partial x_j - \partial u_j/\partial x_i)/2$. The velocity field can be decomposed as

1. A translation $\mathbf{u}(\mathbf{x})$
2. A pure straining flow: $\mathbf{E}$ is a symmetric matrix, with 3 real eigenvalues $\lambda_i$ and 3 associated, mutually orthogonal eigenvectors $\mathbf{p}_i$.

$$tr(\mathbf{E}) = \frac{\partial u_i}{\partial x_i} = \nabla\cdot\mathbf{u} = \sum_i \lambda_i$$

Recall that the trace is invariant under similarity transformations.

$\lambda_i$'s are called the principal rates-of-strain

$\mathbf{p}_i$'s are called the principal axes of strain

Locally

$$\frac{d\mathbf{r}}{dt} = \mathbf{Er} + \mathbf{Wr}$$

Consider first the linear system

$$\dot{\mathbf{r}} = \mathbf{Er} = \mathbf{P\Lambda P}^{-1}\,\mathbf{r}, \text{ with } \Lambda_{ii} = \lambda_i$$

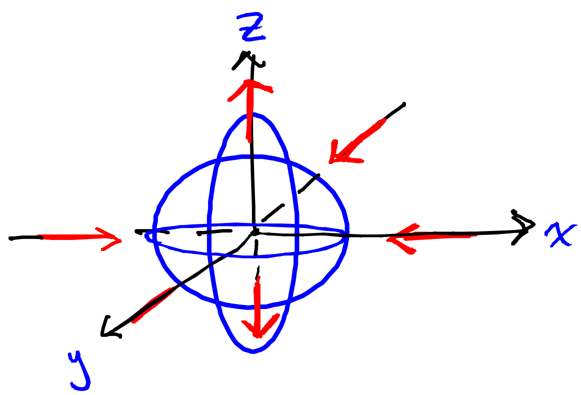
Setting

$$\boldsymbol{\xi} = \mathbf{P}^{-1}\,\mathbf{r} \Rightarrow \dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda\xi}$$

The local effect of **E** is to deform, through compression and expansion, a ball centered at $\mathbf{r} = \mathbf{0}$ into an ellipsoid whose principal axes are the principal axes of strain. The velocity **Er** is called a *pure straining flow*.

Example:

$$\mathbf{E} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$




**An incompressible straining flow**

$$\boldsymbol{\Lambda} = [\boldsymbol{\Lambda} - tr(\boldsymbol{\Lambda})\mathbf{I}/3] + tr(\boldsymbol{\Lambda})\mathbf{I}/3$$
$$= \mathbf{S} + (\nabla \cdot \mathbf{u})_{r=0}\mathbf{I}/3 = \mathbf{S} + \Delta\mathbf{I}/3$$

By construction, $tr(\mathbf{S}) = 0 = \sum_i \lambda_i^S$. Hence the velocity field **Sr** is divergence free and induces no change in volume, while $(\Delta\mathbf{I})\mathbf{r} = \Delta\mathbf{r}$ is a pure (isotropic) compression or expansion.

3. **R** is an anti-symmetric matrix with purely imaginary eigenvalues

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is called the *vorticity*. Vorticity is a fundamental quantity in incompressible fluid dynamics.

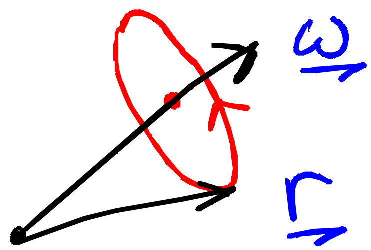
$$\mathbf{W}\mathbf{r} = \frac{1}{2}\begin{pmatrix} \omega_2 r_3 - \omega_3 r_2 \\ \omega_3 r_1 - \omega_1 r_3 \\ \omega_1 r_2 - \omega_2 r_1 \end{pmatrix} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r}$$

The velocity field $\mathbf{W}\mathbf{r}$ is a rigid-body rotation (and is divergence free), w. angular velocity $\frac{1}{2}\boldsymbol{\omega}$. Note

$$\dot{\mathbf{r}} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r} \Rightarrow$$

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 0 \ \text{ fixed length}$$

$$\frac{d}{dt}(\mathbf{r} \cdot \boldsymbol{\omega}) = 0 \ \text{ fixed angle}$$



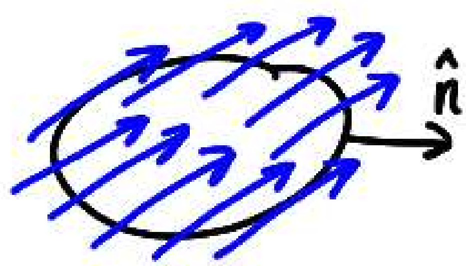

**ω generates a cone upon which r moves**.

*In summary:* The local flow is composed of (i) a translation; (ii) a pure straining flow, itself decomposable into an incompressible part, and an isotropic expansion or compression; (iii) a rigid body rotation.

## Conservation of Mass

Consider a *fixed* subvolume $V_0 \subseteq V$ with outward normal $\hat{\mathbf{n}}$. The mass of $V_0$ at time $t$ is:

$$M[V_0, t] = \int_{V_0} dV_x \rho(\mathbf{x}, t)$$




**The flux of mass through an Eulerian volume $V_0$**

The rate of change of $M[V_0, t]$ is balanced by the flux of mass through its boundary $\partial V_0$, or

$$\frac{d}{dt}\int_{V_0} dV_x \rho = -\int_{\partial V_0} dS_x (\rho\, \mathbf{u}) \cdot \hat{\mathbf{n}} \qquad \#$$

This is the *integral form* of mass conservation. Using the divergence theorem we can write

$$\int_{V_0} dV_x \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho\, \mathbf{u} \right) \right] = 0$$

As $V_0$ was arbitrary, this gives the *continuity equation:*

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla_x \cdot \left( \rho\, \mathbf{u} \right) = 0} \qquad \#$$

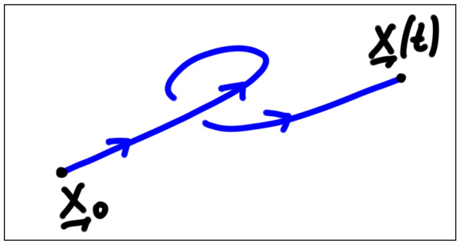
which is a PDE governing the evolution of material density in a moving fluid or continuous material, and is called the *differential form* of mass conservation. $\mathbf{j} = \rho\, \mathbf{u}$ is called the *mass density flux*.

## The Lagrangian formulation

The quantities $\mathbf{u}$ and $\rho$ have been expressed in the *Eulerian frame*, e.g., $\rho$ is measured at a fixed point $\mathbf{x}$. In the *Lagrangian frame* a quantity, say $\rho$, is measured in the frame of moving fluid. Let $\mathbf{X}(t)$ satisfy

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}(t), t) \text{ with } \mathbf{X}(0) = \mathbf{X}_0$$

The function $\mathbf{X}(t)$ is called the *Lagrangian*, *material*, or *particle path.* Consider $\rho(\mathbf{X}(t), t)$, that is, the evolution of fluid density along a Lagrangian path.




**A Lagrangian path**

Then

$$\frac{d}{dt} \rho(\mathbf{X}(t), t) = \left[ \frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \dot{\mathbf{X}} \cdot \nabla_x \rho(\mathbf{x}, t) \right]_{\mathbf{x} = \mathbf{X}(t)}$$

$$= \left[ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla_x \rho \right]_{\mathbf{x} = \mathbf{X}(t)}$$

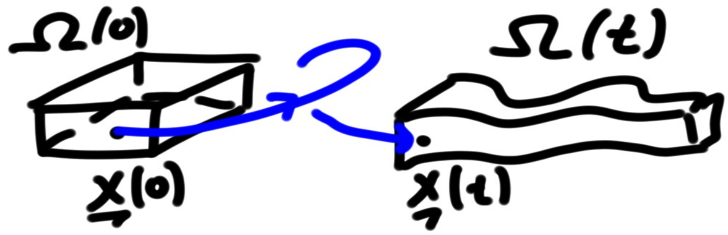
The operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_x$ is called the *Lagrangian, material,* or *substantial derivative*. It is the Eulerian expression for the time-rate-of-change of quantities along Lagrangian paths. And so

$$\boxed{\frac{D\rho}{Dt} = -\rho \left( \nabla_x \cdot \mathbf{u} \right)} \qquad \#$$

*Some properties of the substantial derivative:*

- $\frac{D}{Dt}(fg) = f \frac{Dg}{Dt} + g \frac{Df}{Dt}$ plus other usual aspects of a derivative
- $\frac{Df}{Dt} = 0 \Leftrightarrow f(\mathbf{X}(t), t) = f(\mathbf{X}_0, 0)$, i.e., $f$ is conserved along particle paths.

Previously we had considered a fixed, or Eulerian volume $V_0$. Now, let $\Omega(t)$ be a time dependent volume moved by the flow from $\Omega(0)$:




**The deformation of the Ω under the flow**

That is, solve

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}(t),t) \text{ with } \mathbf{X}(0) = \mathbf{X}_0 \ \forall \mathbf{X}_0 \subseteq \Omega_0$$

$\Omega(t)$ is the set of all consequent $\mathbf{X}(t)$, and is called a *Lagrangian* or *material volume.*

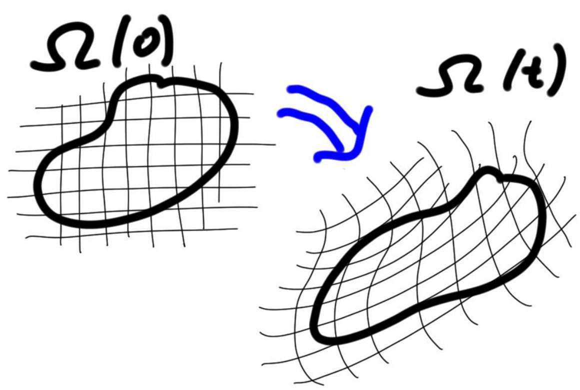
*Lagrangian flow-map:* A *Lagrangian variable* is one that stays constant along a Lagrangian path. The key idea of the Lagrangian formulation is to use the set of initial coordinates $\mathbf{X}_0 \subseteq \Omega_0$ as independent spatial coordinates. So, consider the time-dependent transformation of spatial coordinates

$$\boldsymbol{\alpha} \mapsto \mathbf{X}(\boldsymbol{\alpha},t)$$

found by solving

$$\frac{\partial \mathbf{X}}{\partial t}(\boldsymbol{\alpha},t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha},t),t) \ \text{ with } \ \mathbf{X}(\boldsymbol{\alpha},0) = \boldsymbol{\alpha}$$

(i.e., $\boldsymbol{\alpha} = \mathbf{X}_0$). $\mathbf{X}(\boldsymbol{\alpha},t)$ is the *Lagrangian flow-map* and $\boldsymbol{\alpha}$ is the *Lagrangian variable.*




**The evolution of the Lagrangian flow-map**.

The Lagrangian flow map has many important properties:

**(1)**

$$\begin{aligned}\frac{\partial}{\partial t} f(\mathbf{X}(\boldsymbol{\alpha},t),t) &= \left[\frac{\partial f}{\partial t} + \frac{\partial \mathbf{X}}{\partial t}\cdot\nabla_x f\right]_{\mathbf{x}=\mathbf{X}(\alpha,t)} \\ &= \left[\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla_x f\right]_{\mathbf{x}=\mathbf{X}(\alpha,t)} \\ &= \left[\frac{Df}{Dt}\right]_{\mathbf{x}=\mathbf{X}(\alpha,t)}\end{aligned}$$

Hence, the substantial derivative relates the Eulerian and Lagrangian frames.

(2) A fundamental object defined by the Lagrangian flow-map is the deformation tensor or matrix $\mathbf{F}$, defined as the Jacobian of the flow-map:

$$\mathbf{F} = \frac{\partial \mathbf{X}}{\partial \boldsymbol{\alpha}} \text{ or } F_{ij} = \frac{\partial X_i}{\partial \alpha_j}$$

$F$ encodes the deformations of the Lagrangian flow-map relative to the initial state. Let $\mathbf{V}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. Then $\mathbf{F}$ evolves by

$$\begin{aligned} \frac{\partial F_{ij}}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial X_i}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_j}\frac{\partial X_i}{\partial t} = \frac{\partial V_i}{\partial \alpha_j} \\ &= \frac{\partial}{\partial \alpha_j} u_i(\mathbf{X}(\boldsymbol{\alpha}, t), t) = \frac{\partial u_i}{\partial X_k}\frac{\partial X_k}{\partial \alpha_j} \end{aligned}$$

or

$$\boxed{\frac{\partial \mathbf{F}}{\partial t} = \nabla_\alpha \mathbf{V} = (\nabla_x \mathbf{u})|_{\mathbf{X}}(\boldsymbol{\alpha}, t)\mathbf{F} \text{ with } \mathbf{F}(\boldsymbol{\alpha}, 0) = \mathbf{I}}$$

or in Eulerian variables:

$$\boxed{\frac{D\mathbf{F}}{Dt} = (\nabla_x \mathbf{u})\mathbf{F} \text{ with } \mathbf{F}(\mathbf{x}, 0) = \mathbf{I}}$$

This introduces $\mathbf{D} = \nabla_x \mathbf{u}$, the rate-of-strain tensor. A related tensor is $\mathbf{E} = (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top)/2$, the symmetric rate-of-strain tensor.

(3) Let $J$ be the Jacobian determinant of the flow-map, that is,

$$J(\boldsymbol{\alpha}, t) = \det[\mathbf{F}] = \det\left[\ \mathbf{F}_1, \ldots, \mathbf{F}_n\ \right] = \det\left[\ \nabla_\alpha X_1, \nabla_\alpha X_2, \nabla_\alpha X_3\ \right]$$

*Note:* $J(\boldsymbol{\alpha}, 0) \equiv 1$. We have the following important and standard result from dynamical systems theory for its evolution: *Louiville's Formula:*

$$\boxed{\frac{\partial}{\partial t} J(\boldsymbol{\alpha}, t) = (\nabla_x \cdot \mathbf{u})|_{\mathbf{X}}(\boldsymbol{\alpha}, t)(\mathbf{X}(\boldsymbol{\alpha}, t), t) \cdot J(\boldsymbol{\alpha}, t)}$$

*Proof:* In $\mathbf{R}^n$

$$\mathbf{F} = \left[\nabla_\alpha X_1, \ldots, \nabla_\alpha X_n\ \right] = \left[\ \mathbf{F}_1, \ldots, \mathbf{F}_n\ \right]$$

The Jacobian can be expressed in terms of the multi-linear operator, the *wedge product*:

$$J = \mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \mathbf{F}_n$$

which has the properties:

1. $\mathbf{F}_1 \wedge \ldots \wedge (\alpha \mathbf{U} + \boldsymbol{\beta}\mathbf{W}) \wedge \ldots \wedge \mathbf{F}_n = \alpha(\mathbf{F}_1 \wedge \ldots \wedge \mathbf{U} \wedge \ldots \wedge \mathbf{F}_n) + \beta(\mathbf{F}_1 \wedge \ldots \wedge \mathbf{W} \wedge \ldots \wedge \mathbf{F}_n)$
2. $\mathbf{F}_i \subseteq span[\mathbf{F}_j, j \neq i] \Rightarrow J = 0$
3. $\frac{d}{dt} J = (\dot{\mathbf{F}}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \mathbf{F}_n) + (\mathbf{F}_1 \wedge \dot{\mathbf{F}}_2 \wedge \ldots \wedge \mathbf{F}_n) + \ldots + (\mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \dot{\mathbf{F}}_n)$

Now,

$$\begin{aligned} \mathbf{F}_i &= \frac{\partial}{\partial t}\nabla_\alpha X_i = \nabla_\alpha u_i(\mathbf{X}(\boldsymbol{\alpha}, t), t) \\ &= \frac{\partial u_i}{\partial X_1}(\nabla_\alpha X_1) + \frac{\partial u_i}{\partial X_2}(\nabla_\alpha X_2) + \ldots + \boxed{\frac{\partial u_i}{\partial X_i}(\nabla_\alpha X_i)} + \ldots + \frac{\partial u_i}{\partial X_n}(\nabla_\alpha X_n) \\ &= \frac{\partial u_i}{\partial X_i}\mathbf{F}_i + \mathbf{T}_i \text{ with } \mathbf{T}_i \subseteq span[\mathbf{F}_j, j \neq i] \end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}J &= \left(\left(\frac{\partial u_1}{\partial X_1}\mathbf{F}_1 + \mathbf{T}_1\right) \wedge \mathbf{F}_2 \wedge \ldots \wedge \mathbf{F}_n\right) \\ &+ \left(\mathbf{F}_1 \wedge \left(\frac{\partial u_2}{\partial X_2}\mathbf{F}_2 + \mathbf{T}_2\right) \wedge \ldots \wedge \mathbf{F}_n\right) + \ldots \\ &+ \left(\mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \left(\frac{\partial u_n}{\partial X_n}\mathbf{F}_n + \mathbf{T}_n\right)\right) \\ &= \frac{\partial u_1}{\partial X_1} J + \frac{\partial u_2}{\partial X_2} J + \ldots + \frac{\partial u_n}{\partial X_n} J \\ &= (\nabla_x \cdot \mathbf{u})\, J\end{aligned}$$

(**4**) The effect of change in geometry: Consider two nearby Lagrangian points $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \boldsymbol{\alpha} + d\boldsymbol{\alpha}$. Now consider the displacement of these points in the Eulerian frame under the flow of the material:

$$d\mathbf{X} = \mathbf{X}(\boldsymbol{\beta}, t) - \mathbf{X}(\boldsymbol{\alpha}, t) \approx \mathbf{F}(\boldsymbol{\alpha}, t)d\boldsymbol{\alpha}$$

Then, $|d\mathbf{X}|^2 = d\boldsymbol{\alpha}^T\mathbf{F}^T\mathbf{F}d\boldsymbol{\alpha} = d\boldsymbol{\alpha}^T\mathbf{C}d\boldsymbol{\alpha}$. Hence $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ controls the relative stretching of Lagrangian line elements by the flow. $\mathbf{C}$ is the *right Cauchy-Green tensor*, which is symmetric and positive definite (spd), satisfying $\det\mathbf{C} = (\det\mathbf{F})^2 = J^2 > 0$. $\mathbf{C}$ satisfies the dynamics equation:

$$\mathbf{C}_t = \mathbf{F}_t^T\mathbf{F} + \mathbf{F}^T\mathbf{F}_t = \mathbf{F}^T\,\nabla\mathbf{u}^T\,\mathbf{F} + \mathbf{F}^T\,\nabla_x\mathbf{u}\,\mathbf{F} = 2\mathbf{F}^T\,\mathbf{E}\,\mathbf{F}$$

We can also write $d\boldsymbol{\alpha} = \mathbf{F}^{-1}d\mathbf{X}$, or $|d\boldsymbol{\alpha}|^2 = d\mathbf{X}^T\mathbf{F}^{-T}\mathbf{F}^{-1}d\mathbf{X} = d\mathbf{X}^T(\mathbf{F}\mathbf{F}^T)^{-1}d\mathbf{X} = d\mathbf{X}^T\mathbf{b}^{-1}d\mathbf{X}$. Here

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^T}$$

is the *left Cauchy-Green* (or *Finger*) tensor, which is also spd, and satisfies $\det\mathbf{b} = (\det\mathbf{F})^2 = J^2 > 0$. This tensor arises very naturally in the theory of rubber elasticity. This has the far more attractive dynamics:

$$\frac{D\mathbf{b}}{Dt} = \nabla_x\mathbf{u}\,\mathbf{F}\,\mathbf{F}^T + \mathbf{F}\,\mathbf{F}^T\nabla_x\mathbf{u}^T = \nabla_x\mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x\mathbf{u}^T$$

**Note**: (i) $\mathbf{C}$ and $\mathbf{b}$ have the same invariants. Let $\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ be the singular value decomposition of $\mathbf{F}$, so that $\mathbf{D}$ contains the singular values, and $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices. Then,

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{V}\mathbf{D}^2\mathbf{V}^T \text{ and } \mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{U}\mathbf{D}^2\mathbf{U}^T$$

and so $\mathbf{C}$ and $\mathbf{b}$ have the same eigenvalues, $\lambda_i^2$, and hence have the same invariants. (ii) The evolution for $\mathbf{b}$ is closed, and does not require knowledge of $\mathbf{F}$. This is not so for $\mathbf{C}$.

**Side Note**: The operator $\mathbf{b}^{\nabla} = \frac{D\mathbf{b}}{Dt} - \left(\nabla_x\mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x\mathbf{u}^T\right)$ is called the *upper convected derivative* and is intimately related to conservation principles in the Lagrangian frame. First, a simple proof of the result of Cauchy. For the incompressible 3D Euler equations, vorticity transport is given by (in the Lagrangian frame)

$$\boldsymbol{\omega}_t = \nabla_x\mathbf{u}\,\boldsymbol{\omega} = \nabla_x\mathbf{u}\,\mathbf{F}\mathbf{F}^{-1}\,\boldsymbol{\omega} = \mathbf{F}_t\,\mathbf{F}^{-1}\boldsymbol{\omega} = -\mathbf{F}(\mathbf{F}^{-1})_t\,\boldsymbol{\omega}$$

or

$$\left(\mathbf{F}^{-1}\,\boldsymbol{\omega}\right)_t = \mathbf{0} \text{ giving } \boldsymbol{\omega} = \mathbf{F}\,\boldsymbol{\omega}_0$$

which is the result. Hence, any vector or matrix satisfying

$$\mathbf{W}_t = \nabla_x\mathbf{u}\,\mathbf{W} \text{ also satisfies } \mathbf{W} = \mathbf{F}\,\mathbf{W}_0$$

Consider now the dyadic matrix $\mathbf{Z} = \mathbf{W}\mathbf{W}^T = \mathbf{F}\,\mathbf{W}_0\mathbf{W}_0^T\mathbf{F}^T$. Then

$$\mathbf{Z}_t = \nabla_x\mathbf{u}\,\mathbf{Z} + \mathbf{Z}\,\nabla_x\mathbf{u}^T$$

*In general*, we have the result that $\mathbf{Z}$ satisfies the conservation law

$$(\mathbf{F}^{-1}\mathbf{Z}\mathbf{F}^{-T})_t = \mathbf{0} \text{ or } \mathbf{Z} = \mathbf{F}\,\mathbf{Z}_0\mathbf{F}^T$$

if and only if

$$\mathbf{Z}^{\nabla} \equiv \mathbf{Z}_t - \left(\nabla_x\mathbf{u}\,\mathbf{Z} + \mathbf{Z}\,\nabla_x\mathbf{u}^T\right) = \mathbf{0}$$

## Mass Conservation in the Lagrangian frame

The mass of a Lagrangian volume does not change in time, that is

$$M[\Omega(t)] = M[\Omega(0)]$$

where $M$ can be expressed as:

$$\begin{aligned} M[\Omega(t)] &= \int_{\Omega(t)} dV_x\ \rho(\mathbf{x},t) \text{ tf. to Lagrangian coords} \\ &= \int_{\Omega(0)} dV_\alpha\ \rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t) \end{aligned}$$

Then

$$0 = M[\Omega(t)] - M[\Omega(0)] = \int_{\Omega(0)} dV_\alpha\ [\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t) - \rho(\boldsymbol{\alpha},0)]$$

As $\Omega(0)$ was arbitrary we have

$$\boxed{\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t) = \rho(\boldsymbol{\alpha},0)} \qquad \#$$

This is one Lagrangian form of mass conservation. As $J$ also represents the measure of an infinitesmal volume, it says that if the volume increases, then the density must decrease. This now gives sense to incompressibility of a fluid or material. Incompressibility means that material volumes, infinitesimal or otherwise, do not change their volume. That is, $J(\boldsymbol{\alpha},t) \equiv 1$, and consequently $\rho(\mathbf{X}(\boldsymbol{\alpha},t),t) = \rho(\boldsymbol{\alpha},0)$. This has two consequences, following Liouville's formula:

$$\boxed{(\nabla_x \cdot \mathbf{u}) = 0 \text{ and } \tfrac{D\rho}{Dt} = 0}$$

Hence, incompressible fluids are have divergence free velocity fields, and the density is conserved along Lagrangian paths.

To recover the Eulerian form, take a time-derivative of Eq. (1''') and use the relation with the substantial derivative and Liouville's formula:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t)) \\ &= \left[\frac{D\rho}{Dt}(\mathbf{X}(\boldsymbol{\alpha},t),t)\,J(\boldsymbol{\alpha},t) + \rho(\mathbf{X}(\boldsymbol{\alpha},t),t)(\nabla_x \cdot \mathbf{u})(\mathbf{X}(\boldsymbol{\alpha},t),t)\,J(\boldsymbol{\alpha},t)\right] \\ &= \left[\frac{D\rho}{Dt}(\mathbf{X}(\boldsymbol{\alpha},t),t) + \rho(\mathbf{X}(\boldsymbol{\alpha},t),t)(\nabla_x \cdot \mathbf{u})(\mathbf{X}(\boldsymbol{\alpha},t),t)\right]J(\boldsymbol{\alpha},t) \end{aligned}$$

If the flow is smooth, then $J \neq 0$, and we have

$$\frac{D\rho}{Dt}(\mathbf{x},t) = -\rho(\mathbf{x},t)(\nabla_x \cdot \mathbf{u})(\mathbf{x},t)$$

which we have already proved.

The Lagrangian statement of mass conservation yields the following fundamental result:

*The Transport Theorem:* For any smooth $f(\mathbf{x},t)$

$$\boxed{\frac{d}{dt}\int_{\Omega}(t)dV_x\rho f = \int_{\Omega}(t)dV_x\ \rho\frac{Df}{Dt}}$$

*Proof:* Use that $\partial_t(\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t)) = 0$ and that $\partial_t(f(\mathbf{X}(\boldsymbol{\alpha},t),t)) = (Df/Dt)(\mathbf{X}(\boldsymbol{\alpha},t),t)$:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} dV_x\rho f &= \frac{d}{dt}\int_{\Omega(0)} dV_\alpha\, J\rho f = \int_{\Omega(0)} dV_\alpha\,\frac{\partial}{\partial t}(J\rho f)\\ &= \int_{\Omega(0)} dV_\alpha\left[f\frac{\partial}{\partial t}(J\rho) + (J\rho)\frac{\partial f}{\partial t}\right]\\ &= \int_{\Omega(0)} dV_\alpha\, J\rho\frac{\partial f}{\partial t} = \int_{\Omega(t)} dV_x\,\rho\frac{Df}{Dt}\end{aligned}$$

*Side Note:* If the flow is incompressible, then the deformation tensor $\mathbf{F}$ satisfies $\nabla\cdot\mathbf{F}^T = \mathbf{0}$ at all times.

*Proof:* In Eulerian coordinates, using that $\partial u_i/\partial x_i = 0$, $\mathbf{F}$ satisfies

$$\partial_t F_{ij} + u_k\frac{\partial F_{ij}}{\partial x_k} = \frac{\partial u_i}{\partial x_k}F_{kj}$$

$$\Rightarrow \partial_t\frac{\partial F_{ij}}{\partial x_i} + \underline{\frac{\partial u_k}{\partial x_i}\frac{\partial F_{ij}}{\partial x_k}} + u_k\frac{\partial}{\partial x_k}\frac{\partial F_{ij}}{\partial x_i} = \underline{\frac{\partial u_i}{\partial x_k}\frac{\partial F_{kj}}{\partial x_i}}$$

The underlined terms are identical under interchange of $k$ and $i$. Hence

$$\partial_t\frac{\partial F_{ij}}{\partial x_i} + u_k\frac{\partial}{\partial x_k}\frac{\partial F_{ij}}{\partial x_i} = 0 \text{ or}$$

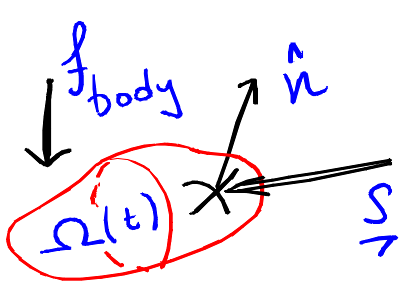
$$\frac{D}{Dt}(\nabla\cdot\mathbf{F}^T) = \mathbf{0} \text{ where } \mathbf{F}_0 = \mathbf{I}$$

## Balance of momentum and forces in a fluid or deformable material

The acceleration of a fluid particle is given by

$$\begin{aligned}\mathbf{a}(t) &= \frac{d^2}{dt^2}\mathbf{X}(t) = \frac{d}{dt}\mathbf{u}(\mathbf{X}(t),t)\\ &= \left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_x)\mathbf{u}\right) = \left(\frac{D\mathbf{u}}{Dt}\right)(\mathbf{X}(t),t)\end{aligned}$$

where notationally $[(\mathbf{u}\cdot\nabla_x)\mathbf{u}]_i = u_j\partial_{x_j}u_i$.




**A Lagrangian subvolume Ω being acted upon by body forces ($\mathbf{f}_{body}$) and forces of stress $(\mathbf{s}\,\Delta A)$.**

Now, let's develop Newton's 2$^{nd}$ Law for balance of forces in a fluid. The momentum carried by a Lagrangian volume of fluid $\Omega(t)$ is

$$\mathbf{m}(t) = \int_{\Omega(t)} dV_x \, \rho\mathbf{u}$$

Generally, forces come in two flavors, *body* and *stress:*

- *Body forces* – externally imposed forces such as gravity or electro-magnetic fields, that exert a force/unit mass. Let $\mathbf{g}(\mathbf{x}, t)$ be such a force/unit mass. The total body force exerted upon $\Omega(t)$ is:

$$\mathbf{f}_{body} = \int_{\Omega(t)} dV_x \, \rho\mathbf{g}$$

- *Forces of stress* – Forces arising from the mechanical contact of the volume $\Omega$, across $\partial\Omega$, with the rest of the fluid or material. According to Cauchy, the (Cauchy) stress $\mathbf{s}$ (units of force/unit area) across a surface with outward normal $\hat{\mathbf{n}}$, at a point $\mathbf{x}$, has the form

$$\mathbf{s} = \boldsymbol{\sigma}\hat{\mathbf{n}} \text{ or } s_i = \sigma_{ij} n_j$$

  $\boldsymbol{\sigma}$ is called the *Cauchy stress tensor* and it is a central focus of nearly all modeling of complex fluids and deformable materials. Conservation of angular momentum implies that the stress tensor is symmetric. The total force of stress exerted upon $\Omega(t)$ is:

$$\mathbf{f}_{stress} = \int_{\partial\Omega(t)} dS_x \, \boldsymbol{\sigma}\hat{\mathbf{n}}$$

Newton's $2^{nd}$ law then gives

$$\frac{d}{dt}\mathbf{m} = \mathbf{f}_{body} + \mathbf{f}_{stress}$$

Applying the transport theorem to the expression for $(d/dt)\mathbf{m}$ and the divergence theorem to the expression of $\mathbf{f}_{stress}$ gives:

$$\int_{\Omega(t)} dV_x \, \rho \frac{Du_i}{Dt} = \int_{\Omega(t)} dV_x \, \rho g_i + \int_{\Omega(t)} dV_x \, \frac{\partial}{\partial x_j}\sigma_{ij}$$

or

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = \nabla_x \cdot \sigma + \rho\mathbf{g}}$$

Write

$$\sigma = -p\mathbf{I} + \mathbf{d} \text{ with } tr(\mathbf{d}) = 0$$

*Newtonian fluids* are those that have a linear relation between the *deviatoric stress* $\mathbf{d}$ and the rate-of-strain tensor $\nabla\mathbf{u}$. All others are termed non-Newtonian.

*Classical examples – (1) the Euler equations.* $\mathbf{d} = \mathbf{0}$. Take the stress to be only in the direction of the normal, that is:

$$\boldsymbol{\sigma} = -p(\mathbf{x}, t)\mathbf{I}$$

$p$ is called the (*mechanical) pressure*, and is compressive for $p > 0$. Hence,

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \rho\mathbf{g}} \quad \text{(L. Euler, 1755)}$$

When the fluid is incompressible, then we have a closed set of evolution equations

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \rho\mathbf{g}$$

$$\frac{D\rho}{Dt} = 0 \text{ and } \nabla_x \cdot \mathbf{u} = 0$$

*Very Important Note:* Here the pressure plays the role of a Lagrange multiplier that enforces incompressibility, adjusting itself at each time to ensure that velocity remains divergence free. This system is *nonlinear* due to the nature of the substantial derivative, but also *nonlocal* as the divergence free condition yields an elliptic character to the equations.

*(2) the isotropic Navier-Stokes equations for an incompressible fluid.*

$$\boldsymbol{\sigma} = -\,p(\mathbf{x},t)\mathbf{I} + 2\mu\mathbf{E}$$

yields the N-S equations

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla_x p + \mu\Delta\mathbf{u} + \rho\mathbf{g}$$

$$\frac{D\rho}{Dt} = 0 \text{ and } \nabla_x \cdot \mathbf{u} = 0$$

$\mu$ is *bulk* or *shear viscosity.*

*(3) a different example: Neo-Hookean elastic solid* – $\boldsymbol{\sigma} = -\,p\mathbf{I} + GJ^{-1}\mathbf{b}$ gives the simplest model of a perfectly elastic solid that dissipates no energy. If the material is incompressible, then when combined with

$$\frac{D\mathbf{b}}{Dt} = \nabla_x\mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x\mathbf{u}^T \text{ and } \nabla_x \cdot \mathbf{u} = 0 \;\; (J \equiv 1)$$

this system is closed.

## Momentum balance in the Lagrangian frame

What is the analogous expression for momentum balance in the Lagrangian frame? For this, we need *Nanson's formula.* This crucial identity allows a change of surface variables between Eulerian and Lagrangian descriptions (Holzapfel, Eq. 2.55):

$$\hat{\mathbf{n}}\,dS_x = J\,\mathbf{F}^{-T}\,\hat{\mathbf{N}}\,dS_\alpha$$

Here $\hat{\mathbf{n}}$ is the normal to a patch of surface of area $dS_x$ in the Eulerian frame, while $\hat{\mathbf{N}}$ is the surface normal to the originating Lagrangian surface of size $dS_\alpha$. Now, here is the proof *not* given by Holzapfel of Nanson's equality in differential form:

$$\boxed{\nabla_x \cdot \boldsymbol{\sigma} = J^{-}1\nabla_\alpha \cdot (\boldsymbol{\sigma}\cdot\mathbf{F}^{-}TJ)}$$

*Proof:* Consider the (stress) tensor $\boldsymbol{\sigma}$ as a set of vectors indexed by $i$, the row index. i.e., $\sigma_{ij} \to \sigma^i_j$. Then, $\nabla_x \cdot \boldsymbol{\sigma}^i = \frac{\partial\sigma^i_j}{\partial x_j}$. Now

$$\frac{\partial}{\partial\alpha_p}\sigma^i_j = \frac{\partial\sigma^i_j}{\partial x_q}\frac{\partial x_q}{\partial\alpha_p} = \frac{\partial\sigma^i_j}{\partial x_q}F_{qp} \text{ or } \nabla_\alpha\boldsymbol{\sigma}^i = \nabla_x\boldsymbol{\sigma}^i\cdot\mathbf{F}$$

$$\text{and hence, } \nabla_x\boldsymbol{\sigma}^i = \nabla_\alpha\boldsymbol{\sigma}^i\cdot\mathbf{F}^{-1} \text{ or taking a trace: } \frac{\partial\sigma^i_j}{\partial x_j} = \frac{\partial\sigma^i_j}{\partial\alpha_p}F^{-1}_{pj}$$

We now make use of the following identity:

$$\frac{\partial}{\partial\alpha_p}(F^{-1}_{pj}J) = [\nabla_\alpha\cdot(\mathbf{F}^{-T}J)]_j = 0$$

(Do this as an exercise.) And so,

$$[\nabla_x\cdot\boldsymbol{\sigma}]_i = J^{-1}\frac{\partial}{\partial\alpha_p}(\sigma_{ij}F^{-1}_{pj}J) = J^{-1}\frac{\partial}{\partial\alpha_p}(\sigma_{ij}F^{-T}_{jp}J) = J^{-1}[\nabla_\alpha\cdot(\boldsymbol{\sigma}\cdot\mathbf{F}^{-T}J)]_i$$

Hence, we have

$$\boxed{\int_{\Omega}(t)\nabla_x \cdot \boldsymbol{\sigma}\, dV_x = \int_{\Omega}(0)\nabla_\alpha \cdot (\boldsymbol{\sigma} \cdot \mathbf{F}^{-}TJ)\, dV_\alpha}$$

or setting $\mathbf{P} = \boldsymbol{\sigma} \cdot \mathbf{F}^{-T}J \Leftrightarrow \boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T$ we have

$$\int_{\Omega(t)} \nabla_x \cdot \boldsymbol{\sigma}\, dV_x = \int_{\Omega(0)} \nabla_\alpha \cdot \mathbf{P}\, dV_\alpha \text{ or}$$

Here $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor, and $\mathbf{P}$ is called the *first Piola-Kirchoff* stress tensor. Since $\boldsymbol{\sigma}$ is symmetric,

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$$

The tensor $\mathbf{P}$ is itself not generally symmetric. We also have

$$\int_{\partial\boldsymbol{\Omega}(\mathbf{t})} \boldsymbol{\sigma}\, \hat{\mathbf{n}}\, dS_x = \int_{\partial\boldsymbol{\Omega}(\mathbf{0})} \mathbf{P}\, \hat{\mathbf{N}}\, dS_\alpha$$

which defines the two stress vectors:

$$\mathbf{s}(\mathbf{x}, t, \hat{\mathbf{n}}) = \boldsymbol{\sigma}(\mathbf{x}, t)\hat{\mathbf{n}} \text{ and } \mathbf{S}(\boldsymbol{\alpha}, t, \hat{\mathbf{N}}) = \mathbf{P}(\boldsymbol{\alpha}, t)\hat{\mathbf{N}}$$

where $\mathbf{s}$ is the (Cauchy) stress relative to the current configuration, and $\mathbf{S}$ is the (first Piola-Kirchoff) stress relative to the reference configuration.

Now, reconsidering balance of momentum,

$$\frac{d}{dt}\int_{\Omega(t)} dV_x \rho(\mathbf{x}, t)\, \mathbf{u}(\mathbf{x}, t) = \int_{\partial\Omega(t)} dS_x \boldsymbol{\sigma}(\mathbf{x}, t)\, \hat{\mathbf{n}} + \int_{\Omega(t)} dV_x \rho(\mathbf{x}, t)\mathbf{g}(\mathbf{x}, t)$$

or, rewriting everything in Lagrangian variables, i.e. letting $\mathbf{V}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$, $\mathbf{G}(\boldsymbol{\alpha}, t) = \mathbf{g}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$, and using $\rho(\boldsymbol{\alpha}, t)\, J(\boldsymbol{\alpha}, t) = \rho_0(\boldsymbol{\alpha})$:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(0)} dV_\alpha \rho(\boldsymbol{\alpha}, t)\, \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)\, J(\boldsymbol{\alpha}, t) &= \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \frac{\partial \mathbf{V}}{\partial t}(\boldsymbol{\alpha}, t)\\ &= \int_{\partial\Omega(0)} dS_\alpha\, \mathbf{P}(\boldsymbol{\alpha}, t)\, \hat{\mathbf{N}}(\boldsymbol{\alpha}) + \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \mathbf{G}(\boldsymbol{\alpha}, t)\end{aligned}$$

And applying the divergence theorem yields:

$$\int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \mathbf{V}_t(\boldsymbol{\alpha}, t) = \int_{\Omega(0)} dV_\alpha\, \nabla_\alpha \cdot \mathbf{P}(\boldsymbol{\alpha}, t) + \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\mathbf{G}(\boldsymbol{\alpha}, t)$$

Now using the arbitrariness of $\Omega(0)$, we have

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = \nabla_\alpha \cdot \mathbf{P} + \rho_0 \mathbf{G}$$

Very nice.

And so, back to the Neo-Hookean solid:

$$\boldsymbol{\sigma} = -p\mathbf{I} + GJ^{-1}\mathbf{b} \Leftrightarrow \mathbf{P} = -p\mathbf{F}^{-T} + G\mathbf{F}$$

Ignoring incompressibility for the moment, in the Lagrangian frame this yields:

$$\begin{aligned}\rho_0 \frac{\partial \mathbf{V}}{\partial t} &= G\, \nabla_\alpha \cdot \mathbf{F}\\ \frac{\partial \mathbf{F}}{\partial t} &= \nabla_a \mathbf{V}\end{aligned}$$

that is, two coupled *linear* PDEs.

For an incompressible material ($J \equiv 1$), we would have:

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla_x p + G\nabla_x \cdot \mathbf{b}$$
$$\frac{D\mathbf{b}}{Dt} = \nabla_x \mathbf{u}\, \mathbf{b} + \mathbf{b}\, \nabla_x \mathbf{u}^T$$
$$\nabla \cdot \mathbf{u} = 0$$

For small displacements: $\mathbf{u} \to \varepsilon \mathbf{u}$, $\mathbf{b} = \mathbf{I} + \varepsilon \mathbf{c}$, and expanding to first-order in $\varepsilon$:

$$\rho_0 \mathbf{u}_t = -\nabla_x p + G\, \nabla_x \cdot \mathbf{c}$$
$$\mathbf{c}_t = \nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T = 2\mathbf{E}$$
$$\nabla \cdot \mathbf{u} = 0$$

Taking a time derivative of the first equation and setting $q = p_t$, we have

$$\rho_0 \mathbf{u}_{tt} = -\nabla_x q + G\, \nabla_x \cdot (\nabla_x \mathbf{u})$$
$$= -\, \nabla_x q + G\Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

That is, an "incompressible" wave equation.

## Conservation of Energy

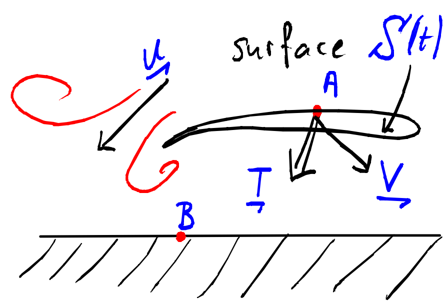
We will come back to this if necessary. Usually associated with non-isothermal situations, which get quite ugly.

# Back to the Navier-Stokes Eqs and its properties.

Let us assume constant density $\rho$, so that

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \mu \Delta \mathbf{u} + \rho \mathbf{g}$$
$$\nabla_x \cdot \mathbf{u} = 0$$

**1**. Boundary conditions on the N-S equations:




**A body of time-dependent (surface** $S(t)$**)**

**moves through a fluid above a solid wall**.

- On a solid boundary, as at point $B$ above, we require for a viscous fluid the *no-slip condition:* $\mathbf{u}|_B = \mathbf{0}$. For an inviscid fluid, $\mathbf{u}|_B \cdot \hat{\mathbf{n}}_{wall} = 0$ so that no fluid penetrates the wall.
- On an impenetrable time-dependent body with surface $S(t)$ which has velocity $\mathbf{V}$ and which exerts a stress $\mathbf{T}$ on the fluid, we require that $\mathbf{u}|_{B\in S} = \mathbf{V}$ and $\boldsymbol{\sigma}|_{B\in S}\, \hat{\mathbf{n}} = -\mathbf{T}$.

- At an interface $S(t)$ between two fluids, or at least two continuum materials, with stress tensors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, we require $\boldsymbol{\sigma}_1\,\hat{\mathbf{n}} - \boldsymbol{\sigma}_2\,\hat{\mathbf{n}} = \mathbf{T}$ where $\mathbf{T}$ is the surface traction. Typically, $\mathbf{T} = \boldsymbol{\gamma\kappa}\hat{\mathbf{n}}$ for surface tension.

2. The N-S equations have a symmetric stress tensor: $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. This guarantees conservation of angular momentum.
3. If no work is done on the system, then N-S has a decaying kinetic energy: Let $\Omega$ be either a fixed closed domain upon whose boundaries the no-slip condition is applied, or all of $R^3$. The kinetic energy is given by

$$K = \frac{1}{2}\int_\Omega \rho\mathbf{u}^2 dV_x$$

and satisfies

$$\dot{K} = \underbrace{-\mu\int_\Omega |\nabla\mathbf{u}|^2 dV_x}_{\text{viscous dissipation to heat}} + \underbrace{\int_\Omega \rho\mathbf{g}\cdot\mathbf{u}}_{\text{work done on the system}}$$

The latter term is zero if the body force arises from a potential. Work can also be done by the time-dependent motion of boundaries in the fluid.
4. The N-S equations are Galilean invariant; that is their form is conserved under the transformation $\mathbf{u} \to \mathbf{u} + \mathbf{U}$ where $\mathbf{U}$ is a constant velocity.
5. Vorticity, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$, is a fundamental quantity for incompressible flows and has distinctly different dynamics in two and three dimensions:

- Vorticity transport and diffusion ($\mathbf{g} = \mathbf{0}$) of 2-d fluid in the $x-y$ plane. Here, vorticity is a scalar $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$. Taking a curl of the momentum equation:

$$\boxed{\frac{D\omega}{Dt} = \nu\Delta\omega}$$

where $\nu = \mu/\rho$ is called the *kinematic viscosity*. This is an advection-diffusion equation.
- In 3-d we have instead:

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = \nabla\mathbf{u}\cdot\boldsymbol{\omega} + \nu\Delta\boldsymbol{\omega}}$$

The extra term, $\nabla\mathbf{u}\cdot\boldsymbol{\omega}$, is the so-called *vorticity stretching term*, and is the term that shows how the vorticity vector field can be amplified or diminished by the local straining flows of the fluid flow, in addition to being advected and diffused. To see this, we recall that $\nabla\mathbf{u} = \mathbf{E} + \mathbf{W}$, where $\mathbf{W}\cdot\mathbf{f} = \boldsymbol{\omega}\times\mathbf{f}$ for any vector $\mathbf{f}$. Hence we have

$$\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{E}\cdot\boldsymbol{\omega} + \nu\Delta\boldsymbol{\omega}$$

Recall that the symmetric rate-of-strain tensor $\mathbf{E}$ is trace-free. If $\boldsymbol{\omega}$ is aligned with a principal direction of positive (negative) rate-of-strain, then the magnitude of $\boldsymbol{\omega}$ will be increased (decreased) (neglecting diffusion). What we shall see shortly is that vorticity actually induces the velocity field, and hence the straining flow in which it evolves. This coupling of vorticity stretching/depletion to the vorticity dynamics itself makes the understanding of the 3-d Navier-Stokes equations especially difficult.
- It is worth examining vorticity transport in the Lagrangian frame in the absence of viscosity. Both equations reflect fundamental conservation laws of the Euler equations. In 2d, the statement $D\omega/Dt = 0$ simply becomes, in the Lagrangian frame:

$$\omega_t = 0 \Rightarrow \boxed{\omega(\boldsymbol{\alpha}, t) = \omega_0(\boldsymbol{\alpha})}$$

or that vorticity is conserved in the Lagrangian frame, that is, along Lagrangian particle paths. The 3d statement is similar, but more complicated. We manipulate the vorticity advection equation in the Lagrangian frame using the evolution equation for the deformation tensor **F**:

$$\boldsymbol{\omega}_t = \nabla_x \mathbf{u} \cdot \boldsymbol{\omega} = (\nabla_x \mathbf{u} \cdot \mathbf{F})\mathbf{F}^{-1} \cdot \boldsymbol{\omega}$$
$$= \mathbf{F}_t \mathbf{F}^{-1} \boldsymbol{\omega} = -\mathbf{F}(\mathbf{F}^{-1})_t \boldsymbol{\omega}$$

giving the conservation law:

$$(\mathbf{F}^{-1}\boldsymbol{\omega})_t = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{\omega}(\boldsymbol{\alpha}, t) = \mathbf{F}(\boldsymbol{\alpha}, t)\boldsymbol{\omega}_0(\boldsymbol{\alpha})} \cdot \cdot$$

This is the so-called *Result of Cauchy*, which states that vorticity is stretched or depleted by the action of the deformation tensor.

**6**. The vorticity-stream formulation establishes the relation between velocity and vorticity.

2D: $\nabla \cdot \mathbf{u} = 0 \Rightarrow \exists\, \psi$ such that $\mathbf{u} = \nabla^{\perp}\psi = (-\psi_y, \psi_x) \Rightarrow \omega = v_x - u_y = \psi_{xx} + \psi_{yy} = \Delta\psi$. For an open flow this then yields the Biot-Savart law.

$$\psi(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{x'} \ln|\mathbf{x} - \mathbf{x}'|\omega(\mathbf{x}') \Rightarrow$$
$$\mathbf{u}(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{x'} \frac{(\mathbf{x} - \mathbf{x}')^{\perp}}{|\mathbf{x} - \mathbf{x}'|}\omega(\mathbf{x}')$$

**7**. In the Lagrangian frame for an inviscid flow we have

$$\mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{\alpha'} \frac{(\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{X}(\boldsymbol{\alpha}', \mathbf{t}))^{\perp}}{|\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{X}(\boldsymbol{\alpha}', t)|}\omega(\mathbf{X}(\boldsymbol{\alpha}', t), t)$$

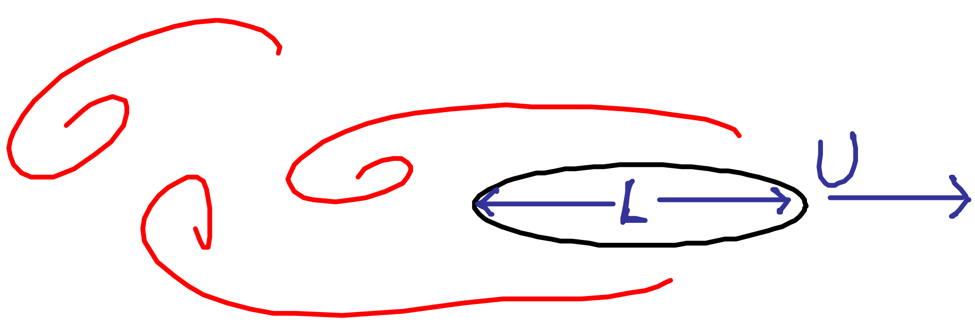
and using the definition of the Lagrangian frame, and conservation of vorticity along particle paths, we have

$$\mathbf{X}_t(\boldsymbol{\alpha}, t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{\alpha'} \frac{(\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{X}(\boldsymbol{\alpha}', \mathbf{t}))^{\perp}}{|\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{X}(\boldsymbol{\alpha}', t)|}\omega_0(\boldsymbol{\alpha}')$$

which is a closed set of equations for the Lagrangian flow map (vorticity moves itself).

**8**. Important, special solutions. In general the nonlinearity of the NS equations, $\mathbf{u} \cdot \nabla\mathbf{u}$, prevents finding analytical solutions, and most know solutions are steady-states for which $\mathbf{u} \cdot \nabla\mathbf{u} = \mathbf{0}$. Most such solutions are *unidirectional flows*.

## The Reynolds Number




The Reynolds number is perhaps the most important dimensionless constant in fluid dynamics. Its magnitude quantifies the relative balance of inertial and viscous forces in a fluid. Consider a body of characteristic size $L$ moving with speed $U$ through a Newtonian fluid. This also defines a characteristic time $T = L/U$. Rescale variables as

$$x \to L\,x,\ \ t \to (L/U)t,\ \ u \to U\,u,\ \text{ and } p \to P\,p$$

Then the incompressible NS eqns become:

$$Re\,\frac{D\mathbf{u}}{Dt} = -\left(\frac{P/L}{\mu U/L^2}\right)\nabla p + \Delta\mathbf{u} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0$$

where all variables are without dimension. Note that the divergence free condition remains unchanged. The dimensionless constant

$$\boxed{Re = \frac{\rho U^2 L^2}{\mu U L} = \frac{\text{"inertial" force}}{\text{"viscous" force}} = \frac{\rho U L}{\mu}}$$

is famous Reynolds number. We have left the pressure scale to be determined. Because of the role the pressure plays in satisfying the divergence free condition it is simply scaled to keep it in the dynamics, regardless of what limiting system in Reynolds number is considered. Consider two extreme, but centrally important cases:

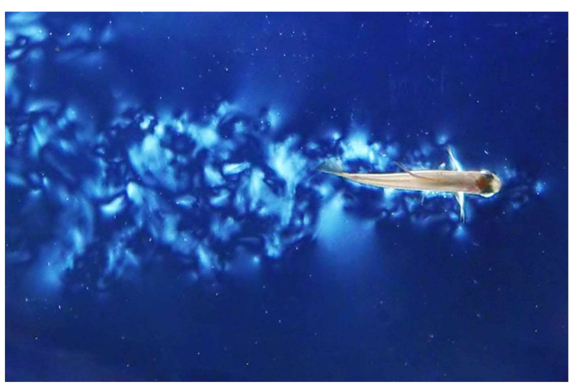
- $Re \gg 1$, meaning that the fluid dynamics is dominated by the inertial forces of the fluid. This is typical for the locomotion of most birds, fish, whales, etc. In this case, choose $P = Re\cdot\mu U/L$, and we have

  $$\frac{D\mathbf{u}}{Dt} = -\nabla p + \frac{1}{Re}\Delta\mathbf{u} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0$$

  Taking the *formal limit* $Re \to \infty$, we get the Euler equations:

  $$\frac{D\mathbf{u}}{Dt} = -\nabla p \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0$$

  We emphasize that this is a formal limit because in the presence of boundaries, static or dynamic, the no-slip condition is a singular perturbation and makes that limit a possibly singular one; There can be a persistent shedding of vorticity produced at the wall even in the limit of infinite Reynolds number. While the Euler equations retain the convective nonlinear of the NS equations, their lack of diffusion gives their dynamics a great deal of geometric structure that is useful in understanding the structure of solutions, as well as giving special tools, such as potential theory, for constructing special classes of solutions.



- $Re \ll 1$, meaning that the fluid dynamics is dominated by the viscous forces of the fluid. This is the typical situation for micro-organismal locomotion, transport of small particles of any sort, and indeed any dynamics that takes place on either a sufficiently slow time-scale, or at a sufficiently small spatial scale. In this case, we choose $P = \mu U/L$, giving

$$Re\frac{D\mathbf{u}}{Dt} = -\nabla p + \Delta\mathbf{u} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0$$

and the formal limit $Re \to 0$ yields the *Stokes equations*:

$$\boxed{-\nabla p + \Delta\mathbf{u} = \mathbf{0} \ \text{ and } \nabla \cdot \mathbf{u} = 0}$$

Note that the Stokes equations are linear, constant coefficient PDEs. For the Stokes equations there is no loss of boundary conditions, unlike the Euler equations, since the highest order spatial term is retained. Unlike either the NS or Euler equations, the Stokes equations are not necessarily solved as an initial value problem as the equations do not contain any time derivatives. They are typically solved as a boundary value problem, where any dynamics devolves from time dependence in boundary data or in solution domain (e.g. as in free boundary problems).

Note that if there free bodies in the fluid, then the low Reynolds number scaling requires that they exert zero net force and torque upon the surrounding fluid. To see this, a body in the fluid moves through Newton's 2nd law as

$$m_b\ddot{\mathbf{X}}_c = \int_\Gamma dS_x(\boldsymbol{\sigma} \cdot \mathbf{n}) \text{ or in dimensionless units}$$

$$\frac{m_b U^2}{L}\ddot{\mathbf{X}}_c = L^2\int_\Gamma dS_x\left(\frac{\mu U}{L}\boldsymbol{\sigma} \cdot \mathbf{n}\right) \text{ which can be rearranged to yield}$$

$$Re\frac{m_b}{m_f}\ddot{\mathbf{X}}_c = \int_\Gamma dS_x(\boldsymbol{\sigma} \cdot \mathbf{n}) \text{ where } m_f = \rho L^3$$

Hence, if $Re \ll 1$, then the inertial term can be dropped so long as $m_b/m_f$ is not large, and we will generate the constraints

$$\mathbf{F} = \int_\Gamma dS_x(\boldsymbol{\sigma} \cdot \mathbf{n}) = \mathbf{0} \text{ and}$$

$$\mathbf{T} = \int_\Gamma dS_x(\mathbf{X} - \mathbf{X}_c) \times (\boldsymbol{\sigma} \cdot \mathbf{n}) = \mathbf{0}$$

- Moderate Reynolds number. In this regime both inertial and viscous forces are important, and this is a regime that has come under increasing scrutiny, for example in studies of small insect locomotion, and efficient mixing in micro-fluidic devices. In the low and high Reynolds regimes there have been many tools – asymptotic reductions, special numerical methods – that have greatly aided in understanding the fluid dynamics. All of these tools fail in the moderate Reynolds number regime, or must be used at best perturbatively, and theoretical studies have been almost exclusively computational in nature.

# The Stokes Equations

The Stokes equations have considerable analytic structure. Again,

$$-\nabla p + \Delta\mathbf{u} = \mathbf{f} \ \text{ and } \ \nabla \cdot \mathbf{u} = 0$$

It is often useful to write them as:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{u} = 0 \ \text{ with } \ \boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{E}$$

Taking a divergence of the momentum equation gives that

$$\Delta p = \nabla \cdot \mathbf{f}$$

so the pressure satisfies a Poisson equation and is harmonic in the absence of an external force. Taking a curl of the momentum equations gives that

$$\Delta\boldsymbol{\omega} = \nabla \times \mathbf{f}$$

so the vorticity also satisfies a Poisson equation. As before, the divergence free condition implies the existence of a vector stream function $\boldsymbol{\Psi}$ which satisfies $\Delta\boldsymbol{\Psi} = -\boldsymbol{\omega}$, and hence

$$\Delta^2\boldsymbol{\Psi} = \nabla \times \mathbf{f}$$

and so the stream-function satisfies a biharmonic equation.

# An Application – Lubrication Theory

Lubrication theory concerns the dynamics of a fluid – a Stokes fluid – in a thin gap. Force balances in such flows are dominated by shear stresses. This arises in many, many instances, such as in the lubrication of joints (a very interesting fluid dynamics problem), or the locomotion of snails and of worms, as well as in many engineering settings such display device design, and scientific problems in pattern formation. The main point here is to derive a simplified version of the Stokes equations that can be more easily analyzed. Consider a long thin channel (in 2d) such as is sketched below, with a horizontal length-scale $L$ and characteristic height $h$, with aspect ratio $\varepsilon = h/L << 1$. We assume the fluid obeys the Stokes equations:

$$\begin{aligned} -p_x + \mu\Delta u &= 0 \\ -p_y + \mu\Delta v &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

We scale each direction on its characteristic length, as well as the associated velocities:

$$x \to Lx;\ y \to hy;\ u \to (L/T)u;\ v \to (L/T)v;\ p \to Pp$$

We assume that there is some characteristic time-scale $T$, perhaps related to an imposed wall velocity, or a force, though this would be given by the precise application. Then, rescaling the (dimensional) Stokes equations we have

$$\begin{aligned} -\frac{PT}{\mu}\varepsilon^2 p_x + (\varepsilon^2 u_{xx} + u_{yy}) &= 0 \\ -\frac{PT}{\mu}p_y + (\varepsilon^2 v_{xx} + v_{yy}) &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

We choose the pressure scale so as to balance pressure stress against the shear stress in the horizontal momentum equation. That is, $P = (\mu/T)\varepsilon^{-2}$ and so

$$\begin{aligned} -p_x + (\varepsilon^2 u_{xx} + u_{yy}) &= 0 \\ -p_y + (\varepsilon^4 v_{xx} + \varepsilon^2 v_{yy}) &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

At leading order we have the *reduced Stokes equations:*

$$\begin{aligned} -p_x + u_{yy} &= 0 \\ p_y &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

and so

$$p = p(x,t)$$
$$\Rightarrow u(x,y,t) = \frac{1}{2}y^2 p_x(x,t) + a(x,t)y$$
that is, the horizontal velocity is a parabolic plus linear shear flow. The vertical velocity is then given by:
$$v(x,y,t) = -\int_0^y \left(\frac{1}{2}y^2 p_{xx} + a_x y\right) dy = -\frac{y^3}{6}p_{xx} + \frac{y^2}{2}a_x$$
There are similar reductions that can be used for non-Newtonian problems involving shear-thinning of -thickening, and elasticity (e.g., see FKSP2001 for its development in non-Newtonian Hele-Shaw flow).

The reduced Stokes equations worked out in 3d leads to Darcy's law for 2d Hele-Shaw flow:
$$\mathbf{v} = -\frac{b^2}{12\mu}\nabla_2 p \text{ and } \nabla_2 \cdot \mathbf{v} = 0$$

# The Stokes solution for a sphere

Consider a sphere of radius $a$ moving at velocity $\mathbf{U} = U\hat{\zeta}$. George Stokes showed that the fluid stress on the sphere is given by
$$\mathbf{\Sigma} = \left[-p_\infty \cos\theta + \mu\frac{3U}{2a}\right]\hat{\zeta}$$
where the polar axis of the sphere is taken in the $\hat{\zeta}$ direction. We then have for the force $\mathbf{F}$ on the sphere the famous Stokes formula:
$$\mathbf{F} = \int_S dA\, \mathbf{\Sigma} = 6\pi\mu a \mathbf{U}$$

# The Jeffrey equation for ellipsoidal particles

Consider an axisymmetric ellipsoid of length $l$ and diameter $d$ rotating in a linear flow $\mathbf{u} = \mathbf{U} + \mathbf{A}\mathbf{x}$ so that $\nabla\mathbf{u} = \mathbf{A} = \mathbf{W} + \mathbf{E}$. Let the unit vector $\mathbf{p}(t)$ point in the direction of the major axix, $\mathbf{X}_c(t)$ be the ellipsoid center, and assume that no force or torque acts upon the ellipsoid. Then (Jeffrey, 1922)
$$\dot{\mathbf{X}}_c = \mathbf{U} + \mathbf{A}\mathbf{X}_c$$
$$\dot{\mathbf{p}} = \mathbf{W}\mathbf{p} + \frac{\lambda^2 - 1}{\lambda^2 + 1}(\mathbf{I} - \mathbf{pp})\mathbf{E}\mathbf{p}$$
$$= (\mathbf{I} - \mathbf{pp})\left[\mathbf{W} + \frac{\lambda^2 - 1}{\lambda^2 + 1}\mathbf{E}\right]\mathbf{p}$$
with $\lambda = l/d$.

- *Sphere:* $\lambda = 1 \Rightarrow \dot{\mathbf{p}} = \mathbf{W}\mathbf{p} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{p}$. Rotation of the director about the vorticity vector. The strain flow contributes nothing to the rotation of the sphere.
- *Slender rod:* $\lambda = \infty \Rightarrow \dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp})\nabla\mathbf{u}\,\mathbf{p}$. Rotated by the flow, but constrained from stretching.
- *Plate:* $\lambda = 0 \Rightarrow \dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp})[\mathbf{W} - \mathbf{E}]\mathbf{p} = -(\mathbf{I} - \mathbf{pp})\nabla\mathbf{u}^T\mathbf{p}$

# Fundamental Solutions of the Stokes Equations

Because the Stokes equations are constant coefficient linear PDEs, solutions to them can be represented in terms of Green's functions. There are several important fundamental solutions for the

Stokes equations, such as the Stokeslet, Rotlet, and Stresslet.

*Formal Construction of the Stokeslet:* Find a solution to the equation

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = -\nabla q + \mu \Delta \mathbf{v} = \hat{\mathbf{e}}\,\delta(\mathbf{x}) \text{ and } \nabla \cdot \mathbf{v} = 0$$

where $\hat{\mathbf{e}}$ is an arbitrary unit vector, and $\delta$ is the 3-d $\delta$-function. Recall that the 3d free-space Green's function for the Laplacian is

$$G = \frac{-1}{4\pi}\frac{1}{|\mathbf{x}|}$$

i.e., $\Delta G = \delta(x)$. Taking a divergence gives $\Delta q = -\hat{\mathbf{e}} \cdot \nabla \delta = -\hat{\mathbf{e}} \cdot \nabla \Delta G = -\Delta(\hat{\mathbf{e}} \cdot \nabla G)$ and so we choose

$$q = -\hat{\mathbf{e}} \cdot \nabla G = \frac{-1}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \hat{\mathbf{e}} = \frac{-1}{4\pi}\frac{\hat{\mathbf{x}}}{|\mathbf{x}|^2} \cdot \hat{\mathbf{e}} = P_k \hat{e}_k$$

Hence, the fundamental solution for the pressure is

$$P_k = \frac{-1}{4\pi}\frac{\hat{x}_k}{|\mathbf{x}|^2}$$

We then have

$$\nabla q = \frac{-1}{4\pi}\frac{1}{|\mathbf{x}|^3}[\mathbf{I} - 3\hat{\mathbf{x}}\hat{\mathbf{x}}]\hat{\mathbf{e}}$$

$$\Rightarrow \mu \Delta \mathbf{v} = \hat{\mathbf{e}}\delta - \nabla(\hat{\mathbf{e}} \cdot \nabla G)$$

Now we construct two functions $B_1,\ B_2$ that satisfy

$$\Delta B_1 = \delta \ \&\ \Delta B_2 = G$$

and let

$$\mathbf{v} = \frac{1}{\mu}[\hat{\mathbf{e}} B_1 - \nabla(\hat{\mathbf{e}} \cdot \nabla B_2)]$$

Now, plucking out only the radially symmetric particular solutions for $B_{1,2}$ gives:

$$B_1 = G \text{ and } B_2 = \frac{-1}{8\pi}|\mathbf{x}|$$

where further calculation gives

$$\mathbf{v} = \frac{-1}{8\pi\mu}\left[\frac{\mathbf{I} + \hat{\mathbf{x}}\hat{\mathbf{x}}}{|\mathbf{x}|}\right]\hat{\mathbf{e}}$$

The rank-two tensor

$$\mathbf{S} = \frac{-1}{8\pi\mu}\left[\frac{\mathbf{I} + \hat{\mathbf{x}}\hat{\mathbf{x}}}{|\mathbf{x}|}\right] \text{ or } S_{ik} = \frac{-1}{8\pi\mu}\left[\frac{\delta_{ik} + \hat{x}_i \hat{x}_k}{|\mathbf{x}|}\right]$$

is called the *Stokeslet* or the *Oseen tensor*. It has a long-range $R^{-1}$ decay and is a negative definite matrix. It can be used to construct other relevant fundamental solutions. We define the *Stresslet* as the rank-three tensor $T_{ijk}$ satisfying $\tilde{\sigma}_{ij} = T_{ijk}\hat{e}_k$, or

$$T_{ijk} = -P_k \delta_{ij} + \mu\left(\frac{\partial S_{ik}}{\partial x_j} + \frac{\partial S_{jk}}{\partial x_i}\right)$$

$$= \frac{3}{4\pi}\frac{\hat{x}_i \hat{x}_j \hat{x}_k}{|\mathbf{x}|^2}$$

The Stokeslet and Stresslet can be used to construct a boundary integral representation for solutions of the Stokes equations, which we very roughly outline (see Pozrikidis for a more detailed derivation). Consider a closed body $B$ with surface $\Gamma$ and outer normal $\hat{\mathbf{n}}$, and with a surface stress distribution $\boldsymbol{\zeta}$ and surface velocity $\mathbf{u}_\Gamma$. Let $(\boldsymbol{\sigma}, \mathbf{u})$ be the Stokes solution that satisfies $\boldsymbol{\sigma}|_\Gamma \hat{\mathbf{n}} = -\boldsymbol{\zeta}$ and

$\mathbf{u}|_\Gamma = \mathbf{u}^\Gamma$.

# The Lorentz Identity

A fundamental identity satisfied by any two solutions $(\boldsymbol{\sigma}, \mathbf{u})$ and $(\tilde{\boldsymbol{\sigma}}, \mathbf{v})$ of the Stokes equation is the Lorentz identity:

$$\nabla \cdot \left(\boldsymbol{\sigma}\,\mathbf{v} - \tilde{\boldsymbol{\sigma}}\mathbf{u}\right) = 0 \ \text{ or } \ \frac{\partial}{\partial x_k}(\sigma_{ki}v_i - \tilde{\sigma}_{ki}u_i) = 0$$

Using symmetry of the stress tensor, we can write:

$$\nabla \cdot \left(\boldsymbol{\sigma}\,\mathbf{v} - \tilde{\boldsymbol{\sigma}}\mathbf{u}\right) = \boldsymbol{\sigma} : \nabla\mathbf{v} - \tilde{\boldsymbol{\sigma}} : \nabla\mathbf{u}$$

$$= [-p\mathbf{I} + 2\mu\mathbf{E}_u]:[\mathbf{E}_v{+}\mathbf{W}_v] - [-q\mathbf{I} + 2\mu\mathbf{E}_v]:[\mathbf{E}_u{+}\mathbf{W}_u] = 0$$

# The Classical Boundary Integral Formulation

We let $(\tilde{\boldsymbol{\sigma}}, \mathbf{v})$ be the Stresslet/Stokeslet pair. Following Pozrikidis, redefine $\mathbf{x} \to \mathbf{x} - \mathbf{y}$, and integrate the above equality over the punctured fluid domain $\Omega / D_\varepsilon(\mathbf{y})$ (with normal into the domain) where $D_\varepsilon(\mathbf{y})$ is the $\varepsilon$-ball about $\mathbf{y}$, hence excluding the singular point from the domain. The divergence theorem then gives

$$0 = \int_{\Gamma + \{|\mathbf{x}-\mathbf{y}|=\varepsilon\}} [v_i(\mathbf{x}'{-}\mathbf{y})\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}') - u_i(\mathbf{x}')\tilde{\sigma}_{ik}(\mathbf{x}'{-}\mathbf{y})n_k(\mathbf{x}')]dA_{x'}$$

(1) On $|\mathbf{x} - \mathbf{y}| = \varepsilon$: Note that on the boundary of $D_\varepsilon(\mathbf{y})$, $\widehat{\mathbf{x}'{-}\mathbf{y}} = \mathbf{n}$,

$$\int_{|\mathbf{x}'-\mathbf{y}|=\varepsilon} S_{ij}(\mathbf{x}'{-}\mathbf{y})\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}')dA_{x'}$$

$$= \frac{-1}{8\pi\mu}\int_{|\mathbf{x}'-\mathbf{y}|=\varepsilon} \frac{\delta_{ij} + n_i n_j}{\varepsilon}\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}')dA_{x'} \to 0 \text{ as } \varepsilon \to 0$$

since the area element scales as $\varepsilon^2$. Now the second term is given by:

$$\frac{3}{4\pi}\int_{|\mathbf{x}-\mathbf{y}|=\varepsilon} u_i \frac{n_i n_j n_k}{\varepsilon^2} n_k dA_{x'} = \frac{3}{4\pi}\varepsilon^{-2}\int_{|\mathbf{x}-\mathbf{y}|=\varepsilon} u_i n_i n_j dA_{x'}$$

$$= \frac{3}{4\pi}\varepsilon^{-2}\varepsilon^2\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \sin\phi (u_i(\mathbf{y}) + O(\varepsilon))n_i(\theta,\phi)n_j(\theta,\phi)$$

$$\approx \frac{3}{4\pi}\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \sin\phi \left[\mathbf{n}\,\mathbf{n}\right]\mathbf{u}(\mathbf{y}) + O(\varepsilon)$$

$$\to \frac{3}{4\pi}\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \sin\phi \begin{bmatrix} \cos^2\theta\sin^2\phi & 0 & 0 \\ 0 & \sin^2\theta\sin^2\phi & 0 \\ 0 & 0 & \cos^2\phi \end{bmatrix}\mathbf{u}(\mathbf{y})$$

$$= u_j(\mathbf{y})$$

One can thus show that:

$$u_j(\mathbf{y}) = -\int_\Gamma [S_{ji}(\mathbf{x} - \mathbf{y})\zeta_i(\mathbf{x}) + u_i(\mathbf{x})T_{ijk}(\mathbf{x} - \mathbf{y})n_k(\mathbf{x})]dA_x$$

or in nicer notation

$$\boxed{\mathbf{u}(\mathbf{y}) = -\int_\Gamma [\mathbf{S}(\mathbf{x} - \mathbf{y})\boldsymbol{\zeta}(\mathbf{x}) + \mathbf{u}^\Gamma(\mathbf{x})\mathbf{T}(\mathbf{x} - \mathbf{y})\hat{\mathbf{n}}(\mathbf{x})]dA_x}$$

Note that $\mathbf{S}$ is even wrt its argument, while $\mathbf{T}$ is odd.

Hence, we have expressed the velocity at every point in the fluid as a function of the surface stress and velocity. Of course the surface velocity and the fluid velocity are related by the no-slip condition, and so it remains to take the limit $\mathbf{y} \to \mathbf{x} \in \Gamma$. The hard one is the stresslet, so let's do that one first. The dominant part of the limit to the surface should arise from this integral:

$$I = \int_{|\mathbf{x}-\mathbf{x}_0|\leq\varepsilon} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{x}) dA_x$$

where $\mathbf{x}_0$ is the closest point to $\mathbf{y}$. Let's replace the $\varepsilon$ –patch with a flat disk, and assume that $\mathbf{y}-\mathbf{x} = r\,\mathbf{n}_0 + \rho\,\mathbf{R}_0\,\mathbf{e}(\theta)$ where $\mathbf{R}_0$ is a rotation matrix ($\mathbf{R}_0\hat{\mathbf{z}} = \mathbf{n}_0$) and $\mathbf{e}(\theta) = (\cos\theta, \sin\theta, 0)$. That is, this is a little $(\rho,\theta)$ coordinate system on the patch. Then

$$\widehat{\mathbf{y}-\mathbf{x}} = \frac{r\,\mathbf{n}_0 + \rho\,\mathbf{R}_0\,\mathbf{e}}{(r^2+\rho^2)^{1/2}}$$

:

$$\begin{aligned}
I &= \frac{-3}{4\pi}\int_0^{2\pi} d\theta \int_0^{\varepsilon} d\rho\,\rho\,\frac{\mathbf{u}\cdot(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})\;\mathbf{n}_0\cdot(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})}{(r^2+\rho^2)^{5/2}}(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e}) \\
&= \frac{-3}{4\pi} r\int_0^{2\pi} d\theta \int_0^{\varepsilon} d\rho\,\rho\,\frac{\mathbf{u}\cdot(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})}{(r^2+\rho^2)^{5/2}}(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e}) \\
&= \frac{-3}{4\pi} r\int_0^{2\pi} d\theta \int_0^{\varepsilon} d\rho\,\rho\,\frac{r^2(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0+\rho^2(\mathbf{u}\cdot\mathbf{R}_0\mathbf{e})\mathbf{R}_0\mathbf{e}}{(r^2+\rho^2)^{5/2}}
\end{aligned}$$

Ok, we need to calculate

$$\begin{aligned}
\int_0^{2\pi} d\theta(\mathbf{u}\cdot\mathbf{R}_0\mathbf{e})\mathbf{R}_0\mathbf{e} &= \mathbf{R}_0\left(\int_0^{2\pi} d\theta\,\mathbf{e}\mathbf{e}\right)\mathbf{R}_0^T\mathbf{u} \\
&= \pi\mathbf{R}_0(\mathbf{I}-\hat{\mathbf{z}}\hat{\mathbf{z}}^T)\mathbf{R}_0^T\mathbf{u} \\
&= \pi(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)
\end{aligned}$$

And so

$$\begin{aligned}
I &= \frac{-3}{4} r\int_0^{\varepsilon} d\rho\,\rho\,\frac{2r^2(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0+\rho^2(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)}{(r^2+\rho^2)^{5/2}} \\
&= \frac{-3}{2}\int_0^{\varepsilon} d(\rho/r)\,\frac{(\rho/r)}{\left(1+(\rho/r)^2\right)^{5/2}}(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 - \frac{3}{4}\int_0^{\varepsilon} d(\rho/r)\frac{(\rho/r)^3}{\left(1+(\rho/r)^2\right)^{5/2}}(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0) \\
&= \frac{-3}{2}\int_0^{\delta}\frac{x}{(1+x^2)^{5/2}}dx\,(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 - \frac{3}{4}\int_0^{\delta}\frac{x^3}{(1+x^2)^{5/2}}dx\,(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)\;;\;\delta = \varepsilon/r \\
&= \frac{-1}{2}\frac{(\delta^2+1)^{\frac{3}{2}}-1}{(\delta^2+1)^{\frac{3}{2}}}(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 + \frac{1}{4}\frac{1}{(\delta^2+1)^{\frac{3}{2}}}\left(3\delta^2-2(\delta^2+1)^{\frac{3}{2}}+2\right)(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)
\end{aligned}$$

Now, we need to take $r \to 0$ for $\varepsilon$ fixed, that is, $\delta \to \infty$. This yields

$$I = \frac{-1}{2}(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 - \frac{1}{2}(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0) = -\frac{1}{2}\mathbf{u}$$

And so, in this limit we have

$$u_j(\mathbf{x}) = -\int_\Gamma S_{ji}(\mathbf{x}'-\mathbf{x})\zeta_i(\mathbf{x}')dA_{x'}$$

$$\frac{1}{2}u_j(\mathbf{x}) - P\int_\Gamma u_i(\mathbf{x}')T_{ijk}(\mathbf{x}'-\mathbf{x})n_k(\mathbf{x}')dA_{x'}$$

or

$$\boxed{\tfrac{1}{2}\mathbf{u}^\Gamma(\mathbf{x}) + P\int_\Gamma \mathbf{u}^\Gamma(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_x' = -\int_\Gamma \mathbf{S}(\mathbf{x}'-\mathbf{x})\boldsymbol{\zeta}(\mathbf{x}')dS_{x}/}$$

or a Fredholm integral equation of the second kind for the surface velocity, or a first-kind equation for the surface stress. If we are given the surface stress, then this equation can in principal be solved for the surface velocity, and then used to give the fluid velocity everywhere in the fluid domain. This is one of the fundamental relations of the Stokes equations.

Now, for illustration, consider a rigid body moving under an applied force $\mathbf{F}$ and torque $\mathbf{L}$, that is

$$\int_\Gamma dS_x\, \boldsymbol{\zeta}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) = \mathbf{F} \text{ and } \int_\Gamma dS_x(\mathbf{x}-\mathbf{X}_c)\times\boldsymbol{\zeta}(\mathbf{x}) = \mathbf{L}$$

and being rigid means that for the surface velocity $\mathbf{u}^\Gamma = \mathbf{U} + (\mathbf{x}-\mathbf{X}_c(t))\times\boldsymbol{\Omega}(t)$. Before inserting this into the integral equation we note two identities for $\mathbf{x} \in \Omega$ (the fluid domain):

$$P\int_\Gamma \mathbf{v}\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = \mathbf{0} \text{ for any constant vector } \mathbf{v}.$$

$$P\int_\Gamma x_l' T_{ijk}(\mathbf{x}'-\mathbf{x})n_k(\mathbf{x}')dS_{x'} = 0$$

or

$$\boxed{\mathbf{u}(\mathbf{x}) = -\int_\Gamma \mathbf{S}(\mathbf{x}'-\mathbf{x})\boldsymbol{\zeta}(\mathbf{x}')dS_{x}/}$$

which means that for rigid bodies, taking the limit $\mathbf{x}\to\Gamma$, we have:

$$\boxed{\mathbf{U} + (\mathbf{x}-\mathbf{X}_c(t))\times\boldsymbol{\Omega}(t) = -\int_\Gamma \mathbf{S}(\mathbf{x}'-\mathbf{x})\boldsymbol{\zeta}(\mathbf{x}')dS_{x}/}$$

which is an integral equation for $\boldsymbol{\zeta}$ in terms of the two unknowns $\mathbf{U}$ and $\boldsymbol{\Omega}$. The system is closed by the specification of the force and torque. The body is then evolved via

$$\dot{\mathbf{X}}_c = \mathbf{U} \text{ and } \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}$$

Note however that this is essentially a first-kind integral equation for the surface stress $\boldsymbol{\zeta}$, and is widely used but ill-conditioned. Need an argument for this...

## The formulation of Power & Miranda

PM1987 use an Ansatz of a velocity induced by a distribution of stresslets:

$$\mathbf{u}(\mathbf{x}) = \int_\Gamma \boldsymbol{\psi}(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} + \ldots$$

This is incomplete however because this flow induces no force or torque upon the body, i.e.,

$$\int_\Gamma \boldsymbol{\sigma}_{\mathbf{u}}(\mathbf{x}')\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = \mathbf{0} \text{ and } \int_\Gamma (\mathbf{x}'-\mathbf{X}_c)\times\boldsymbol{\sigma}_{\mathbf{u}}(\mathbf{x}')\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = \mathbf{0}$$

Indeed, in the 2nd-kind integral equation there is a rank-two deficiency. To corrct this, one must add explicit flow contributions from a Stokeslet (which generates a unit-scale force) and a Rotlet (which generates a unit-scale torque). Hence,

$$\mathbf{u}(\mathbf{x}) = \int_\Gamma \boldsymbol{\psi}(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} + \mathbf{S}(\mathbf{x}-\mathbf{X}_c)\mathbf{F} + \mathbf{R}(\mathbf{x}-\mathbf{X}_c)\mathbf{L}$$

together with the conditions that relate $\boldsymbol{\psi}$ to $\mathbf{F}$ and $\mathbf{L}$:

$$\int_{\Gamma} \boldsymbol{\psi}(\mathbf{x})dS_x = \mathbf{F} \text{ and } \int_{\Gamma} (\mathbf{x}' - \mathbf{X}_c) \times \boldsymbol{\psi}(\mathbf{x})dS_x = \mathbf{0}$$

which identifies $\boldsymbol{\psi}$ as a force density. The extra terms are known as the completion flow.

Taking $\mathbf{x} \to \Gamma$ then gives

$$\mathbf{U} + (\mathbf{x} - \mathbf{X}_c(t)) \times \boldsymbol{\Omega}(t) = -\frac{1}{2}\boldsymbol{\psi}(\mathbf{x}) + P\int_{\Gamma} \boldsymbol{\psi}(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} + \mathbf{S}(\mathbf{x} - \mathbf{X}_c)\mathbf{F} + \mathbf{R}(\mathbf{x} - \mathbf{X}_c)\mathbf{L}$$

which is a well-conditioned, full-rank system for $\boldsymbol{\psi}$, $\mathbf{U}$, and $\boldsymbol{\Omega}$.

To solve this problem, the whole game is the numerical quadrature of the singular integral contribution. Recall that $\mathbf{T}$ is odd, with a $|\mathbf{x}|^{-2}$ singularity, and the integral is apparently of principal value type. However, let's introduce surface coordinates $(\alpha, \beta)$, and so write

$$P\int_{\Gamma} \boldsymbol{\psi}(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = P\int_{\Gamma} \boldsymbol{\psi}(\alpha', \beta')\mathbf{T}(\mathbf{x}(\alpha', \beta')-\mathbf{x}(\alpha, \beta))\hat{\mathbf{n}}(\alpha', \beta')J_{\alpha\beta}(\alpha', \beta')dS_{\alpha\beta}$$

Then, wlog setting $(\alpha, \beta) = (0, 0)$, we have in the neighborhood of the singularity that

$$\mathbf{x}(\alpha', \beta')-\mathbf{x}(0,0) \approx \frac{\partial \mathbf{x}}{\partial \alpha}(0,0)\alpha' + \frac{\partial \mathbf{x}}{\partial \beta}(0,0)\beta' + HOTs$$

Both $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ are both tangent to $\Gamma$, and hence orthogonal to $\mathbf{n}$. This means that the PV integral actually has a singularity of first-order, not second, which is integrable. Still, before proceeding we recall another identity:

$$P\int_{\Gamma} \bar{\boldsymbol{\psi}}\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = \frac{1}{2}\bar{\boldsymbol{\psi}} \text{ for any constant } \bar{\boldsymbol{\psi}}.$$

Hence

$$-\frac{1}{2}\boldsymbol{\psi}(\mathbf{x}) + P\int_{\Gamma} \boldsymbol{\psi}(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = \int_{\Gamma} [\boldsymbol{\psi}(\mathbf{x}') - \boldsymbol{\psi}(\mathbf{x})]\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'}$$

which has no divergence at all, which is not to say that it is smooth. The integrand is actually bounded but multi-valued at the origin, with a value depending upon the direction of approach. This is the convenient form for numerical integration. An easy approach is the so-called *Point Vortex Method*, which comes from the vorticity formulation of the 2D Euler equations. The Biot-Savart integral has a similar structure.

$$\mathbf{U} + (\mathbf{x} - \mathbf{X}_c(t)) \times \boldsymbol{\Omega}(t) = -\frac{1}{2}\boldsymbol{\psi}(\mathbf{x}) + \int_{\Gamma} [\boldsymbol{\psi}(\mathbf{x}') - \boldsymbol{\psi}(\mathbf{x})]\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} + \mathbf{S}(\mathbf{x} - \mathbf{X}_c)\mathbf{F} + \mathbf{R}(\mathbf{x} - \mathbf{X}_c)\mathbf{L}$$

Reintroduce the surface coordinates, discretize uniformly in them (i.e., generalized spherical surface coordinates), and use collocation:

$$\int_{\bar{\Gamma}} [\boldsymbol{\psi}(\alpha', \beta') - \boldsymbol{\psi}(\alpha_j, \beta_j)]\mathbf{T}(\mathbf{x}(\alpha', \beta')-\mathbf{x}(\alpha_j, \beta_j))\hat{\mathbf{n}}(\alpha', \beta')J_{\alpha\beta}(\alpha', \beta')dS_{\alpha'\beta'} = \int_{\bar{\Gamma}} I(\alpha', \beta'; \alpha_j, \beta_j)dS_{\alpha\beta}$$

which is ill-defined for $(\alpha', \beta') = (\alpha_j, \beta_j)$. Approximate the integral simply by omitting the singular point in the evaluation:

$$\int_{\bar{\Gamma}} I(\alpha', \beta'; \alpha_j, \beta_j)dS_{\alpha\beta} \approx h_\alpha h_\beta \sum_{\substack{k,l \\ (k,l)\neq(i,j)}} v_k^\alpha v_l^\beta I(\alpha_k, \beta_l; \alpha_j, \beta_j)$$

where $v_k^\alpha$ and $v_l^\beta$ are quadrature weights. This will yield second-order accuracy (KS2011) using the trapezoidal rule. The integral relations of $\boldsymbol{\psi}$ to $\mathbf{F}$ and $\mathbf{L}$ are nonsingular.

So, the set of equations have the form:

$$\mathbf{U} + (\mathbf{x}_{ij} - \mathbf{X}_c) \times \mathbf{\Omega} - h_\alpha h_\beta \sum_{\substack{k,l \\ (k,l)\neq(i,j)}} v_k^\alpha v_l^\beta I(\alpha_k, \beta_l; \alpha_j, \beta_j) = \mathbf{S}(\mathbf{x}_{ij} - \mathbf{X}_c)\mathbf{F} + \mathbf{R}(\mathbf{x}_{ij} - \mathbf{X}_c)\mathbf{L}$$

$$h_\alpha h_\beta \sum_{k,l} v_k^\alpha v_l^\beta \boldsymbol{\psi}_{kl} = \mathbf{F}$$

$$h_\alpha h_\beta \sum_{k,l} v_k^\alpha v_l^\beta (\mathbf{x}_{kl} - \mathbf{X}_c) \times \boldsymbol{\psi}_{kl} = \mathbf{L}$$

This is a large dense system of equations, $\mathbf{Az} = \mathbf{b}$, for $N = 3(N_\alpha \times N_\beta) + 6$ unknowns. Not so bad to solve directly for a single body, but for many it becomes prohibitive. Instead one uses an iterative scheme, such as GMRES, that only requires matrix multiplies. Matrix multiplies require $O(N^2)$ floating point operations. This can be reduced to $O(N)$ using methods such as FMM (Greengard et al), or kernel-independent FMM (Biros, Zorin, et al). Effective preconditioning can key to finding an solution, accurate to a specified tolerance, in a number of steps that is $N$ independent (or at least weakly so).

Comments:

1. Is easily reformulated for solving for forces and torques, given the rigid body motion (NRZS2016).
2. Close interactions of bodies is problematic. This is being overcome (slowly) through the development of QBX schemes.
3. If one needs to know the surface stresses, Keaveny & Shelley (2011) have developed a 2nd-kind integral equation formulation based on the Power-Miranda formulation. It is only applicable to rigid body motion (new Spagnolie work), and was used as the basis for shape optimization studies of magnetically driven microswimmers (KSW2013). Also, the completion flow is not unique, and they show that different choices can lead to markedly improved numerical results.
4. Coupling of many bodies, and to background flows, is straightforward.

# Slender-Body Theory

See Tornberg & Shelley (J. Comp. Phys. 196, 8-40 (2004); TS2004) for discussion and references (most especially Keller & Rubinow ( JFM 1976), Johnson (JFM 1980), and Gotz (PhD thesis 2000)).

## The dynamics of a small rigid rod in a background flow

For a rigid rod we can write $\mathbf{X}(s,t) = \mathbf{X}_c(t) + s\mathbf{p}(t)$. Assume that the length of the rod is very small relative to the length-scale of the flow, i.e., $\mathbf{U}(\mathbf{x}) \approx \mathbf{U}(\mathbf{X}_c) + \nabla\mathbf{U}(\mathbf{X}_c)(\mathbf{x} - \mathbf{X}_c)$

$$\eta\left[\dot{\mathbf{X}}_c - \mathbf{U}(\mathbf{X}_c) + s\left(\dot{\mathbf{p}} - \nabla\mathbf{U}(\mathbf{X}_c)\,\mathbf{p}\right)\right] = (\mathbf{I} + \mathbf{p}\mathbf{p}^T)\mathbf{f}$$

$$\text{with } \int_{-L/2}^{L/2} ds\, \mathbf{f}(s) = \mathbf{0} \text{ and}$$

$$\int_{-L/2}^{L/2} ds\, (\mathbf{X}(s,t) - \mathbf{X}_c(t)) \times \mathbf{f}(s) = \mathbf{p} \times \int_{-L/2}^{L/2} ds\, s\mathbf{f}(s) = \mathbf{0}$$

we have

$$\mathbf{f} = \eta\left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)\left[\dot{\mathbf{X}}_c - \mathbf{U}(\mathbf{X}_c) + s\left(\dot{\mathbf{p}} - \nabla\mathbf{U}(\mathbf{X}_c)\,\mathbf{p}\right)\right]$$

and from the force-free condition and oddness of $\mathbf{f}$ gives

$$\dot{\mathbf{X}}_c(t) = \mathbf{U}(\mathbf{X}_c) \text{ and hence } \mathbf{f} = \eta s\left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)\left(\dot{\mathbf{p}} - \nabla\mathbf{U}(\mathbf{X}_c)\,\mathbf{p}\right)$$

Zero torque gives:

$$\mathbf{p} \times \left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)\left(\dot{\mathbf{p}} - \nabla\mathbf{U}(\mathbf{X}_c)\,\mathbf{p}\right) = \mathbf{0} \Leftrightarrow$$
$$\mathbf{p} \times \left(\dot{\mathbf{p}} - \nabla\mathbf{U}(\mathbf{X}_c)\,\mathbf{p}\right) = \mathbf{0}$$

Now we use that $\mathbf{p} \times \mathbf{p} \times \mathbf{g} = -(\mathbf{I} - \mathbf{p}\mathbf{p})\mathbf{g}$ and that $\mathbf{p} \cdot \dot{\mathbf{p}} = 0$ to get

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{p}\mathbf{p}^T)\nabla\mathbf{U}(\mathbf{X}_c)\,\mathbf{p}$$

Finally, we calculate the force itself:

$$\mathbf{f} = -s\eta\left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)(\mathbf{p}\mathbf{p}^T : \nabla\mathbf{U})\mathbf{p} = -\frac{s\eta}{2}(\mathbf{p}\mathbf{p}^T : \nabla\mathbf{U})\mathbf{p}$$

## Applied to Simple Swimmers

Here given in 2d, proceeds as follows: Consider a inextensible swimmer of finite length $L$, with its time-dependent shape given by $\theta_s = \kappa(s,t)$, the curvature. We represent the body as

$$\mathbf{X}(s,t) = \tilde{\mathbf{X}}(t) + I_0[\mathbf{X}_s](s,t)$$
$$\theta(s,t) = \tilde{\theta}(t) + I_0[\kappa](s,t)$$

where

$$I_0[f](s) = \int_0^s f\,ds' - \frac{1}{L}\int_0^L ds\int_0^s f\,ds'$$
$$\Rightarrow \langle I_0[f]\rangle \equiv \frac{1}{L}\int_0^L I_0[f]\,ds = 0$$

so that we care dealing naturally with centroidal coordinates. Then

$$\mathbf{X}_t = \tilde{\mathbf{X}}_t(t) + I_0\left[\mathbf{X}_s^\perp\,\theta_t\right] \;\&\; \theta_t = \tilde{\theta}_t(t) + I_0[\kappa_t] \Rightarrow$$
$$\mathbf{X}_t = \tilde{\mathbf{X}}_t(t) + I_0\left[\mathbf{X}_s^\perp\,(\tilde{\theta}_t(t) + I[\kappa_t])\right]$$
$$= \tilde{\mathbf{X}}_t(t) + \tilde{\theta}_t(t)I_0[\mathbf{X}_s^\perp] + I_0\left[\mathbf{X}_s^\perp\,I_0[\kappa_t]\right]$$
$$\mathbf{f} = \frac{\mu}{2}\mathbf{D}\left(\tilde{\mathbf{X}}_t(t) + \tilde{\theta}_t(t)I_0[\mathbf{X}_s^\perp] + I_0\left[\mathbf{X}_s^\perp\,I_0[\kappa_t]\right]\right)$$

and finally:

$$\left(\int_0^L ds\,\mathbf{D}(s)\right)\tilde{\mathbf{X}}_t(t) + \tilde{\theta}_t(t)\int_0^L ds\,\mathbf{D}(s)\,I_0[\mathbf{X}_s^\perp](s)$$
$$= -\int_0^L ds\,\mathbf{D}(s)\,I_0\left[\mathbf{X}_s^\perp\,I_0[\kappa_t]\right](s)$$
$$\left(\int_0^L ds\,\mathbf{X}^\perp(s)\cdot\mathbf{D}(s)\right)\tilde{\mathbf{X}}_t(t) + \tilde{\theta}_t(t)\int_0^L ds\,\mathbf{X}^\perp(s)\cdot\mathbf{D}(s)\,I_0[\mathbf{X}_s^\perp](s)$$
$$= -\int_0^L ds\,\mathbf{X}^\perp(s)\cdot\mathbf{D}(s)\,I_0\left[\mathbf{X}_s^\perp\,I_0[\kappa_t]\right](s)$$

A couple of comments are in order:

1. First, note that the viscosity does not show up in determining the velocity or the rate-of-rotation. This is typical of Stokes. If the motion of a boundary is specified independently of the viscosity, then the consequent fluid motion will be also independent of it. This is easily seen by noting

that viscosity could be scaled out of Stokes by rescaling the pressure. However, this is no longer true if boundary forces that are irreversible, rather than boundary position, is specified.

2. This analysis is easy to replicate for 3d motions, where a rotation matrix $\mathbf{R}$ with two degrees of freedom replaces the angle $\theta$.
3. What happens if the swimmer is executing only small amplitude motions, that is, $\kappa = \varepsilon\eta$, with $\varepsilon << 1$. From the presence of $\kappa_t$ on the RHS of the equations we see immediately that the RHS is $O(\varepsilon)$. We note that $\zeta = I_0[\eta]$ is the only term appearing. Wlog we assume that $\tilde{\mathbf{X}}_t$ and $\tilde{\theta}_t$ are zero. Then

$$\begin{aligned}
\theta &= \varepsilon\zeta \\
\mathbf{X}_s &= \left(1 - \frac{1}{2}\varepsilon^2\zeta^2 + O(\varepsilon^4), \varepsilon\zeta + O(\varepsilon^3)\right) \\
\mathbf{X}_s^{\perp} &= \left(-\varepsilon\zeta + O(\varepsilon^3), 1 - \frac{1}{2}\varepsilon^2\zeta^2 + O(\varepsilon^4)\right) \\
\mathbf{X} &= \left((s - L/2) - \frac{1}{2}\varepsilon^2 I_0[\zeta^2] + O(\varepsilon^4), \varepsilon I_0[\zeta] + O(\varepsilon^3)\right)
\end{aligned}$$

and so

$$\mathbf{D} = \mathbf{I} + \mathbf{X}_s^{\perp}\mathbf{X}_s^{\perp} = \begin{pmatrix} (1 + \varepsilon^2\zeta^2) + O(\varepsilon^4) & -\varepsilon\zeta + O(\varepsilon^3) \\ -\varepsilon\zeta + O(\varepsilon^3) & (2 - \varepsilon^2\zeta^2) + O(\varepsilon^4) \end{pmatrix}$$

$$\Rightarrow \langle \mathbf{D} \rangle = \begin{pmatrix} 1 + O(\varepsilon^2) & O(\varepsilon^3) \\ O(\varepsilon^3) & 2 + O(\varepsilon^2) \end{pmatrix}$$

$$\begin{aligned}
I_0[\mathbf{X}_s^{\perp}] &= \mathbf{X}^{\perp} \text{ (in this instance)} \\
&= \left(-\varepsilon I_0[\zeta] + O(\varepsilon^3), (s - L/2) - \frac{1}{2}\varepsilon^2 I_0[\zeta^2] + O(\varepsilon^4)\right) \\
I_0\left[\mathbf{X}_s^{\perp}\, I_0[\kappa_t]\right] &= \varepsilon I_0\left[\zeta_t\left(-\varepsilon\zeta + O(\varepsilon^3), 1 - \frac{1}{2}\varepsilon^2\zeta^2 + O(\varepsilon^4)\right)\right] \\
&= \varepsilon\partial_t\left(-\frac{1}{2}\varepsilon I_0[\zeta^2] + O(\varepsilon^3), I_0[\zeta] - \frac{1}{6}\varepsilon^2 I_0[\zeta^3] + O(\varepsilon^4)\right)
\end{aligned}$$

and so again:

$$\begin{aligned}
&\mathbf{D}\, I_0[\mathbf{X}_s^{\perp}] \\
&= \begin{pmatrix} (1 + \varepsilon^2\zeta^2) + O(\varepsilon^4) & -\varepsilon\zeta + O(\varepsilon^3) \\ -\varepsilon\zeta + O(\varepsilon^3) & (2 - \varepsilon^2\zeta^2) + O(\varepsilon^4) \end{pmatrix}\begin{pmatrix} -\varepsilon I_0[\zeta] + O(\varepsilon^3) \\ (s - L/2) - \frac{1}{2}\varepsilon^2 I_0[\zeta^2] + O(\varepsilon^4) \end{pmatrix} \\
&= \begin{pmatrix} -\varepsilon(I_0[\zeta] + \zeta(s - L/2)) + O(\varepsilon^3) \\ 2(s - L/2) + O(\varepsilon^2) \end{pmatrix} \\
&\Rightarrow \left\langle \mathbf{D}\, I_0[\mathbf{X}_s^{\perp}] \right\rangle = \begin{pmatrix} -\varepsilon\langle\zeta(s - L/2)\rangle + O(\varepsilon^3) \\ O(\varepsilon^2) \end{pmatrix}
\end{aligned}$$

$$\mathbf{D}\, I_0\left[\mathbf{X}_s^\perp\, I_0[\kappa_t]\right]$$
$$= \varepsilon\begin{pmatrix} (1+\varepsilon^2\zeta^2)+O(\varepsilon^4) & -\varepsilon\zeta+O(\varepsilon^3) \\ -\varepsilon\zeta+O(\varepsilon^3) & (2-\varepsilon^2\zeta^2)+O(\varepsilon^4) \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\varepsilon\partial_t I_0[\zeta^2]+O(\varepsilon^3) \\ I_0[\zeta_t]-\frac{1}{6}\varepsilon^2\partial_t I_0[\zeta^3]+O(\varepsilon^4) \end{pmatrix}$$
$$= \varepsilon\begin{pmatrix} -\varepsilon\left(\frac{1}{2}\partial_t I_0[\zeta^2]+\zeta I_0[\zeta_t]\right)+O(\varepsilon^3) \\ 2I_0[\zeta_t]+O(\varepsilon^2) \end{pmatrix}$$
$$\Rightarrow \left\langle \mathbf{D}\, I_0\left[\mathbf{X}_s^\perp\, I_0[\kappa_t]\right]\right\rangle = \begin{pmatrix} -\varepsilon^2\langle\zeta I_0[\zeta_t]\rangle+O(\varepsilon^4) \\ O(\varepsilon^3) \end{pmatrix}$$

Note that since $\langle\zeta\rangle = 0,$ we can write $\zeta = p_s$ where $p|_{s=0,L} = 0$. Then, $\langle\zeta(s-L/2)\rangle = -\langle p\rangle$The first equation then becomes

$$\begin{pmatrix} 1+O(\varepsilon^2) & O(\varepsilon^3) \\ O(\varepsilon^3) & 2+O(\varepsilon^2) \end{pmatrix}\begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix}+\begin{pmatrix} -\varepsilon\langle p\rangle+O(\varepsilon^3) \\ O(\varepsilon^2) \end{pmatrix}\tilde{\theta}_t$$
$$= \begin{pmatrix} -\varepsilon^2\langle\zeta I_0[\zeta_t]\rangle+O(\varepsilon^4) \\ O(\varepsilon^3) \end{pmatrix}$$

The only leading order behaviors that are consistent with this equation are that $\tilde{x}_t = O(\varepsilon^2)$, $\tilde{y}_t = O(\varepsilon^3)$, and $\tilde{\theta}_t = O(\varepsilon)$, that is, that the speed of the swimmer increases only quadratically with amplitude of deformation. This is consistent with Taylor's results for a swimming sheet in a Stokes fluid. This calculation needs to be finished.

While slightly tiresome, I would like to rewrite all of the "swimming" material in a form invariant under the precise position and rotation. Define

$$\boldsymbol{R}=\begin{pmatrix} \cos\tilde{\theta} & -\sin\tilde{\theta} \\ \sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix};$$
$$\nu = \theta-\tilde{\theta} = I_0[\kappa];\ \ \mathbf{X}(s,t)-\tilde{\mathbf{X}}(t) = \boldsymbol{R}\,\mathbf{Y}$$
$$\mathbf{Y}_s = (\cos\nu,\sin\nu) \Rightarrow \mathbf{X}_s = \boldsymbol{R}\,\mathbf{Y}_s\ \ \&\ \ \mathbf{X}_s^\perp = \boldsymbol{R}\,\mathbf{Y}_s^\perp$$
$$\tilde{\mathbf{X}}_t = \boldsymbol{R}\,\tilde{\mathbf{Y}}_t \Rightarrow$$
$$\mathbf{X}_t = \boldsymbol{R}\,\left(\tilde{\mathbf{Y}}_t(t)+\tilde{\theta}_t(t)I_0[\mathbf{Y}_s^\perp]+I_0\left[\mathbf{Y}_s^\perp\, I_0[\kappa_t]\,\right]\right)$$

using that

$$\mathbf{D}= \mathbf{I}+\mathbf{X}_s^\perp\mathbf{X}_s^\perp = \boldsymbol{R}(\mathbf{I}+\mathbf{Y}_s^\perp\mathbf{Y}_s^\perp)\boldsymbol{R}^T = \boldsymbol{R}\mathbf{D}_Y\boldsymbol{R}^T$$
$$\mathbf{f} = \frac{\mu}{2}\boldsymbol{R}\mathbf{D}_Y\left(\tilde{\mathbf{Y}}_t(t)+\tilde{\theta}_t(t)I_0[\mathbf{Y}_s^\perp]+I_0\left[\mathbf{Y}_s^\perp\, I_0[\kappa_t]\,\right]\right) = \frac{\mu}{2}\boldsymbol{R}\,\boldsymbol{g}$$

It is easy to show that

$$\int_0^L \boldsymbol{g} = \mathbf{0}\ \ \text{and}\ \ \int_0^L \mathbf{Y}^\perp\cdot\boldsymbol{g} = 0$$

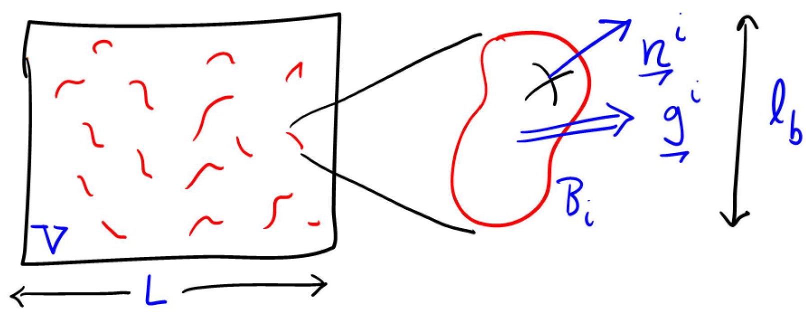
and finally:

$$\left(\int_0^L ds\,\mathbf{D}_Y(s)\right)\tilde{\mathbf{Y}}_t(t) + \tilde{\theta}_t(t)\int_0^L ds\,\mathbf{D}_Y(s)\,I_0[\mathbf{Y}_s^\perp](s)$$
$$= -\int_0^L ds\,\mathbf{D}_Y(s)\,I_0\Big[\mathbf{Y}_s^\perp\,I_0[\kappa_t]\Big](s)$$
$$\left(\int_0^L ds\,\mathbf{Y}^\perp(s)\cdot\mathbf{D}_Y(s)\right)\tilde{\mathbf{Y}}_t + \tilde{\theta}_t(t)\int_0^L ds\,\mathbf{Y}^\perp(s)\cdot\mathbf{D}_Y(s)\,I_0[\mathbf{Y}_s^\perp](s)$$
$$= -\int_0^L ds\,\mathbf{Y}^\perp(s)\cdot\mathbf{D}_Y(s)\,I_0\Big[\mathbf{Y}_s^\perp\,I_0[\kappa_t]\Big](s)$$

In this form everything is expressed in terms of purely geometric quantities.

## Deriving the Batchelor formula

Consider a system volume $\mathbf{V}$ of volume $L^3$ and containing $N$ particles of length-scale $l_b$. Assume that $\mathbf{V}$ can be parcellated into many subvolumes of length-scale $l$. We will make some separation of scale arguments when we need them, such as $l_b << l << L$.




We would like to calculate the total average stress in a volume containing a Newtonian liquid in which are immersed many small bodies, where each body exerts a stress $\mathbf{g}^i$ on the surrounding fluid. Center the volume on point $\mathbf{x}$ and write $\Omega[\mathbf{x}]$. Let the fluid subdomain be $\tilde{\Omega}_f$ and the particle subdomain be $\Omega_p = \cup_n B_n$. We assume that both fluid and particle is described by the zero divergence stress tensors $\boldsymbol{\sigma}^f$ and $\boldsymbol{\sigma}^p$, respectively, and require that $\mathbf{g} = -\boldsymbol{\sigma}^p|_{\partial B_i}\mathbf{n} = \boldsymbol{\sigma}^f|_{\partial B_i}\mathbf{n}$. The bodies have outward normals while $\partial\Omega$ has inward.

We write $\boldsymbol{\sigma} = \chi\,\boldsymbol{\sigma}^f + (1-\chi)\boldsymbol{\sigma}^p$ with $\chi$ being the indicator function of $\Omega_f$. The average stress $\bar{\boldsymbol{\sigma}}$ is then:

$$\bar{\boldsymbol{\sigma}}(\mathbf{x}) = \frac{1}{V}\int_{\Omega[\mathbf{x}]} dV_y\,\boldsymbol{\sigma}(\mathbf{y}) = \frac{1}{V}\left[\int_{\Omega_f[\mathbf{x}]} dV_y\,\boldsymbol{\sigma}^f(\mathbf{y}) + \int_{\Omega_p[\mathbf{x}]} dV_y\,\boldsymbol{\sigma}^p(\mathbf{y})\right]$$

Now some work. We make two assumptions for now, first that inertia is unimportant, and second that there is no intersection of an immersed body with $\partial\Omega$.

(**1**) First show that $\nabla\cdot\bar{\boldsymbol{\sigma}} = \mathbf{0}$:

$$\nabla\cdot\int_{\Omega_f[\mathbf{x}]} dV_y\,\boldsymbol{\sigma}^f(\mathbf{y}) = -\int_{\partial\Omega[\mathbf{x}]} dS\,\boldsymbol{\sigma}^f(\mathbf{y})\mathbf{n}(\mathbf{y})$$
$$= \int_{\partial\Omega_p[\mathbf{x}]} dS\,\boldsymbol{\sigma}^f(\mathbf{y})\mathbf{n}(\mathbf{y}) + \int_{\Omega_f[x]} dV\,\nabla\cdot\boldsymbol{\sigma}^f(\mathbf{y})$$

The first term is zero because of the force-free condition on each particle (this could be relaxed), and the second because it just is.

$$\nabla \cdot \int_{\Omega_p[\mathbf{x}]} dV_y\, \boldsymbol{\sigma}^p(\mathbf{y}) = \int_{\partial\Omega_p[\mathbf{x}]} dS\, \boldsymbol{\sigma}^p(\mathbf{y})\mathbf{n}(\mathbf{y}) = \int_{\partial\Omega_p[\mathbf{x}]} dS\, \boldsymbol{\sigma}^f(\mathbf{y})\mathbf{n}(\mathbf{y}) = \mathbf{0}$$

Note that I did not use that $\boldsymbol{\sigma}^p$ is divergence free.

(**2**) The pressure and velocity are only defined in the fluid region, and so I define

$$\bar{\mathbf{u}} = \frac{1}{V_f} \int_{\Omega_f[\mathbf{x}]} dV_y\, \mathbf{u} \Rightarrow$$

$$\begin{aligned}
\nabla \cdot \int_{\Omega_f[\mathbf{x}]} dV_y\, \mathbf{u} &= -\int_{\partial\Omega[\mathbf{x}]} dS\, \mathbf{u} \cdot \mathbf{n} \\
&= \int_{\partial\Omega_p[\mathbf{x}]} dS\, \mathbf{u} \cdot \mathbf{n} + \int_{\Omega_f[x]} dV\, \nabla \cdot \mathbf{u} \\
&= 0
\end{aligned}$$

Here we assume that there is no mass flux from the particles (this could be relaxed to model volume changes and fluid exchange).

(**3**) Now let's average our Newtonian stress tensor:

$$\begin{aligned}
\int_{\Omega_f[\mathbf{x}]} dV_y\, \boldsymbol{\sigma}^f &= \int_{\Omega_f[\mathbf{x}]} dV_y\, [-p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)] \\
&= V_f \left(-\bar{p}\mathbf{I} + \mu\left(\overline{\nabla\mathbf{u}} + \overline{\nabla\mathbf{u}}^T\right)\right) \\
&= V_f\,(-\bar{p}\mathbf{I}) - \mu\left[\int_{\partial\Omega[\mathbf{x}]} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T) + \int_{\partial\Omega_p[\mathbf{x}]} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)\right]
\end{aligned}$$

and now calculate $\nabla\bar{\mathbf{u}}$ to eliminate the outer boundary term:

$$\nabla \int_{\Omega_f[\mathbf{x}]} dV_y\, \mathbf{u} = -\int_{\partial\Omega[\mathbf{x}]} dS\, \mathbf{u}\mathbf{n}^T$$

yielding:

$$\int_{\Omega_f[\mathbf{x}]} dV_y\, \boldsymbol{\sigma}^f = V_f\,(-\bar{p}\mathbf{I} + \mu(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)) - \mu \int_{\partial\Omega_p[\mathbf{x}]} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)$$

Nice.

(**4**) and we average our particle stress tensor, using Batchelor's little trick to express the internal particle stress in terms of surface quantities:

$$\begin{aligned}
\int_{\Omega_p[\mathbf{x}]} dV_y\, \sigma^p_{ij}(\mathbf{y}) &= \int_{\Omega_p[\mathbf{x}]} dV_y \sigma^p_{ik}(\mathbf{y}) \frac{\partial y_j}{\partial y_k} \\
&= \int_{\partial\Omega_p[\mathbf{x}]} dS_y\, \sigma^p_{ik} y_j n_k - \int_{\Omega_p[\mathbf{x}]} dV_y\, y_j \frac{\partial}{\partial y_k} \sigma^p_{ik}
\end{aligned}$$

Now we use that $\nabla \cdot \boldsymbol{\sigma}^p = \mathbf{0}$, and write

$$\int_{\Omega_p[\mathbf{x}]} dV_y\, \boldsymbol{\sigma}^p(\mathbf{y}) = \int_{\partial\Omega_p[\mathbf{x}]} dS_y\, \mathbf{g}\mathbf{y}^T$$

In summary then, we have derived the Kirkwood-Batchelor formula:

$$\nabla \cdot \bar{\boldsymbol{\sigma}} = \mathbf{0}, \ \nabla \cdot \bar{\mathbf{u}} = 0 \text{ with}$$

$$\bar{\boldsymbol{\sigma}} = \frac{V_f}{V}\left[-\bar{p}\mathbf{I} + \mu(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)\right] + \frac{1}{V}\left[\int_{\partial\Omega_p[\mathbf{x}]} dS_y\, \mathbf{g}\mathbf{y}^T - \mu\int_{\partial\Omega_p[\mathbf{x}]} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)\right] \text{ or}$$

$$-\nabla\bar{p} + \Delta\bar{\mathbf{u}} = -\nabla \cdot \boldsymbol{\sigma}^e \quad \text{and} \quad \nabla \cdot \bar{\mathbf{u}} = 0$$

$$\text{with } \boldsymbol{\sigma}^e = -\frac{1}{V_f}\left[\int_{\partial\Omega_p[\mathbf{x}]} dS_y\, \mathbf{g}\mathbf{y}^T - \mu\int_{\partial\Omega_p[\mathbf{x}]} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)\right]$$

# a. Two spheres connected by a spring in a linear background flow.

As a model for a polymer coil, consider two spheres connected by a spring between their centers $\mathbf{X}_1$ and $\mathbf{X}_2$, and moving in a background flow. We assume that the two spheres have no direct hydrodynamic interaction, and only interact through the spring. We then use Stokes' formula to calculate the dynamics of the spheres. Label the spheres 1 and 2, of radius $a$, so that

$$6\pi\mu a(\dot{\mathbf{X}}_1 - \mathbf{u}(\mathbf{X}_1)) = \mathbf{F}(\mathbf{X}_1 - \mathbf{X}_2)$$

$$6\pi\mu a(\dot{\mathbf{X}}_2 - \mathbf{u}(\mathbf{X}_2)) = -\mathbf{F}(\mathbf{X}_1 - \mathbf{X}_2)$$

These two particles comprise a zero force particle, as was assumed for deriving Batchelor's formula. For the distension, or end-to-end displacement, vector $\mathbf{R} = \mathbf{X}_1 - \mathbf{X}_2$ we have

$$6\pi\mu a(\dot{\mathbf{R}}_1 - [\mathbf{u}(\mathbf{X}_1) - \mathbf{u}(\mathbf{X}_2)]) = 2\mathbf{F}(\mathbf{R})$$

Expanding in small displacement about the midpoint $\mathbf{X}_c = (\mathbf{X}_1 + \mathbf{X}_2)/2$

$$6\pi\mu a(\dot{\mathbf{R}} - \nabla_x\mathbf{u}(\mathbf{X}_c)\mathbf{R}) = 2\mathbf{F}(\mathbf{R}) \Rightarrow$$

$$\dot{\mathbf{R}} = \nabla_x\mathbf{u}(\mathbf{X}_c)\mathbf{R} + \frac{1}{3\pi\mu a}\mathbf{F}(\mathbf{R}) \text{ and similarly}$$

$$\dot{\mathbf{X}}_c = \mathbf{u}(\mathbf{X}_c)$$

Note that we have a stretching equation again for $\mathbf{R}$. What is the extra stress produced by this single pair? Let the sphere surface be given by $\mathbf{X}_i + a\mathbf{r}(y)$ with $y$ a surface coordinate on the unit sphere. From Batchelor

$$\boldsymbol{\sigma}^e = -\frac{1}{V_f}\int_{S_1+S_2} dS_x\, [\mathbf{g}\mathbf{x}^T - \mu(\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)] \text{ with } \mathbf{g} = \boldsymbol{\sigma}^f\mathbf{n}$$

the stress exerted by the flow upon the sphere. Assuming that each sphere moves rectilinearly as a rigid body, we have $\mathbf{u}_{1,2} = consts$, and $\mathbf{g}_1 = \frac{3\mu}{2a}\mathbf{F}(\mathbf{R})$ and $\mathbf{g}_2 = -\frac{3\mu}{2a}\mathbf{F}(\mathbf{R})$, so that

$$\boldsymbol{\sigma}^e = -\frac{1}{V_f}\int_S dS_y \begin{bmatrix} \frac{3\mu}{2a}\mathbf{F}(\mathbf{R})(\mathbf{X}_1 + a\mathbf{r}(y))^T - \frac{3\mu}{2a}\mathbf{F}(\mathbf{R})(\mathbf{X}_2 + a\mathbf{r}(y))^T \\ -\mu(\mathbf{u}_1\mathbf{n}^T + \mathbf{n}\mathbf{u}_1^T + \mathbf{u}_2\mathbf{n}^T + \mathbf{n}\mathbf{u}_2^T) \end{bmatrix}$$

$$= -\frac{1}{V_f}\frac{3\mu}{2a}4\pi a^2\mathbf{F}(\mathbf{R})\mathbf{R}^T = -\frac{1}{V_f}6\pi\mu a\, \mathbf{F}(\mathbf{R})\mathbf{R}^T$$

where we used that for any closed surface $\int_S dS\mathbf{n} = \mathbf{0}$. If the spring is a linear Hookean spring, $\mathbf{F}(\mathbf{R}) = -k\mathbf{R}$ we then have

$$\boldsymbol{\sigma}^e = \frac{1}{V_f}6\pi\mu ak\, \mathbf{R}\mathbf{R}^T$$

## b. A rigid fiber in a linear background flow.

## b. A swimming rod in a background flow.

Consider a slender rod $\mathbf{X}(s,t) = \bar{\mathbf{X}}(t) + s\mathbf{p}(t)$ with $-l/2 \le s \le l/2$ where we pose a propulsive surface stress for negative $s$ and a no-slip condition and consequent drag for positive $s$. Slender body theory:

$$\eta[\mathbf{x}_t - \mathbf{u}_\infty] = (\mathbf{I} + \mathbf{ss})\mathbf{f}$$

where $\eta = 8\pi\mu/(-c) > 0$ (following TS2004; $c = \ln\varepsilon^2 e$) and $\mathbf{f}$ is the force/length acting on the fluid by the filament.

(i) No background flow: The first version to consider is the following system:

$$-\,l/2 \le s \le 0:\ \ \eta[U + u_\parallel(s)]\mathbf{p} = (\mathbf{I} + \mathbf{pp})\mathbf{f}_1$$

where $\mathbf{f}_1 = -f_\parallel(s)\mathbf{p}$ with $f_\parallel > 0$ and

$$0 \le s \le l/2:\ \ \eta U\mathbf{p} = (\mathbf{I} + \mathbf{pp})\mathbf{f}_2$$

That is, $U$ is the speed of propagation, $u_\parallel(s)$ is the surface slip, $\mathbf{f}_1$ is the propulsive stress, and $\mathbf{f}_2$ is the drag stress. This is completed by the requirement of zero total force. Note that $\mathbf{f}_2$ must be in the $\mathbf{p}$ direction. Given $f_\parallel$ we then have the three equations

$$\eta[U + u_\parallel] = -2f_\parallel$$

$$\eta U = 2f_2$$

$$-\int_{-l/2}^{0} ds\, f_\parallel + \frac{l}{2}f_2 = 0$$

This set of equations has the solution:

$$f_2 = \frac{1}{l/2}\int_{-l/2}^{0} ds\, f_\parallel$$

$$U = \frac{2}{\eta}\frac{1}{l/2}\int_{-l/2}^{0} ds\, f_\parallel > 0$$

$$u_\parallel = -U - \frac{2}{\eta}f_\parallel = -\frac{4}{\eta l}\int_{-l/2}^{0} ds\, f_\parallel - \frac{2}{\eta}f_\parallel < 0$$

**Example 1**: $f_\parallel$ a constant.

$$f_2 = f_\parallel,\ U = \frac{2}{\eta}f_\parallel,\ u_\parallel = -\frac{4}{\eta}f_\parallel$$

Or, using that $f_\parallel = 2\pi a g_\parallel$ with $g_\parallel$ the surface stress:

$$U = \frac{2}{8\pi\mu/|\ln\varepsilon^2 e|}2\pi a g_\parallel = \frac{\varepsilon|\ln\varepsilon^2 e|}{2}\frac{lg_\parallel}{\mu} = \kappa_2\frac{lg_\parallel}{\mu}$$

Let's try and calculate the extra-stress contributions. From above, it's density has the form:

$$\mathbf{S} = -\int_{-l/2}^{+l/2} ds\, \mathbf{f}(s)\mathbf{x}^T(s) = -\left[\int_{-l/2}^{0} ds(-f_\parallel\mathbf{p})(\bar{\mathbf{X}}(t) + s\mathbf{p}(t))^T + \int_{0}^{l/2} ds(f_2\mathbf{p})(\bar{\mathbf{X}}(t) + s\mathbf{p}(t))^T\right]$$

$$= -\left[\int_{-l/2}^{0} ds\, sf_\parallel\,\mathbf{pp}^T - \int_{0}^{l/2} ds\, sf_\parallel\,\mathbf{pp}^T\right] = \frac{-1}{2}\left[-s^2|_{-l/2}^{0} + s^2|_0^{l/2}\right]f_\parallel\mathbf{pp}^T = \frac{-l^2}{4}f_\parallel\mathbf{pp}^T$$

$$= -\frac{l^2}{4}(2\pi a g_\parallel)\mathbf{pp}^T = -\frac{\pi\varepsilon}{2}l^3 g_\parallel\mathbf{pp}^T = -\kappa_1 l^3 g_\parallel\mathbf{pp}^T$$

Note that $\kappa_{1,2}$ are solely geometric constants.

(ii) With a background flow: Now let $\mathbf{x}(s,t) = \mathbf{x}_0(t) + s\mathbf{p}(t)$ so that $\mathbf{x}_t = \dot{\mathbf{x}}_0 + s\dot{\mathbf{p}}$. Here it will be interesting to make general the sets where different BCs are applied. Let $[-l/2, +l/2] = \Omega_1 + \Omega_2$ where $\Omega_1$ and $\Omega_2$ are disjoint measurable sets (!) with $l_i = meas(\Omega_i)$ and $\chi_i$ their characteristic functions. Hence, we consider

$$\Omega_1:\ \eta[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}} + u_{\|}(s)\mathbf{p} - \mathbf{u}(\mathbf{x}_0 + s\mathbf{p})] = (\mathbf{I} + \mathbf{pp})\mathbf{f}_1$$

$$\text{where } \mathbf{f}_1 = -f_{\|}(s)\mathbf{p} + \mathbf{g} \text{ with } f_{\|} > 0$$

$$\Omega_2:\ \eta[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}} - \mathbf{u}(\mathbf{x}_0 + s\mathbf{p})] = (\mathbf{I} + \mathbf{pp})\mathbf{f}_2$$

Here $\mathbf{g}$ will pick up the part of the stress due to rotations, and $u_{\|}$ picks up the effect of motive stress. Note, this is a choice!

Noting that $\mathbf{f} = \chi_1\mathbf{f}_1 + \chi_2\mathbf{f}_2$ we have:

$$\eta\left[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}} + \chi_1 u_{\|}(s)\mathbf{p} - \mathbf{u}(\mathbf{x}_0 + s\mathbf{p})\right] = (\mathbf{I} + \mathbf{pp})\mathbf{f}$$

The condition of zero force gives

$$\dot{\mathbf{x}}_0 = \frac{1}{l}\int_{-l/2}^{l/2} ds\, \mathbf{u}(\mathbf{x}_0 + s\mathbf{p}) - \frac{1}{l}\int_{\Omega_1} ds\, u_{\|}(s)\mathbf{p}$$

The torque is given by,

$$\int_{-l/2}^{l/2} ds\, s\mathbf{p}\times\mathbf{f} = \eta\mathbf{p}\times\left(\mathbf{I} - \frac{1}{2}\mathbf{pp}\right)\int_{-l/2}^{l/2} ds\left[s^2\dot{\mathbf{p}} + \chi_1 s u_{\|}(s)\mathbf{p} - s\mathbf{u}(\mathbf{x}_0 + s\mathbf{p})\right]$$

Noting that $\mathbf{p}\times\left(\mathbf{I} - \frac{1}{2}\mathbf{pp}\right)\mathbf{g} = \mathbf{p}\times\mathbf{g}$ for any $\mathbf{g}$ gives

$$\int_{-l/2}^{l/2} ds\, s\mathbf{p}\times\mathbf{f} = \eta\mathbf{p}\times\int_{-l/2}^{l/2} ds[s^2\dot{\mathbf{p}} - s\mathbf{u}(\mathbf{x}_0 + s\mathbf{p})]$$

$$= \eta\left[\frac{l^3}{12}\mathbf{p}\times\dot{\mathbf{p}} - \mathbf{p}\times\int_{-l/2}^{l/2} ds\, s\,\mathbf{u}(\mathbf{x}_0 + s\mathbf{p})\right] \Rightarrow$$

$$\mathbf{p}\times\dot{\mathbf{p}} = \frac{12}{l^3}\mathbf{p}\times\int_{-l/2}^{l/2} ds\, s\,\mathbf{u}(\mathbf{x}_0 + s\mathbf{p})$$

Now use that $\mathbf{p}\times(\mathbf{p}\times\mathbf{q}) = -(\mathbf{I} - \mathbf{pp})\mathbf{q}$ and that $(\mathbf{I} - \mathbf{pp})\dot{\mathbf{p}} = \dot{\mathbf{p}}$:

$$\dot{\mathbf{p}} = \frac{12}{l^3}(\mathbf{I} - \mathbf{pp})\int_{-l/2}^{l/2} ds\, s\,\mathbf{u}(\mathbf{x}_0 + s\mathbf{p})$$

This is the generic result since the slip velocity drops out. And so, we have

$$\mathbf{f} = \eta\left(\mathbf{I} - \frac{1}{2}\mathbf{pp}\right)\left[\begin{array}{c} \left(\chi_1 u_{\|}(s) - \frac{1}{l}\int_{\Omega_1} ds\, u_{\|}(s)\right)\mathbf{p} \\ -\left(\mathbf{u}(\mathbf{x}_0 + s\mathbf{p}) - \frac{1}{l}\int_{-l/2}^{l/2} ds\, \mathbf{u}(\mathbf{x}_0 + s\mathbf{p})\right) \\ +s\frac{12}{l^3}(\mathbf{I} - \mathbf{pp})\int_{-l/2}^{l/2} ds\, s\,\mathbf{u}(\mathbf{x}_0 + s\mathbf{p}) \end{array}\right]$$

Or, on $\Omega_1$:

$$-f_\parallel(s)\mathbf{p}+\mathbf{g}=\eta\left(\mathbf{I}-\frac{1}{2}\mathbf{pp}\right)\left[\begin{array}{c}\left(u_\parallel(s)-\frac{1}{l}\int_{\Omega_1}ds\,u_\parallel(s)\right)\mathbf{p}\\ -\left(\mathbf{u}(\mathbf{x}_0+s\mathbf{p})-\frac{1}{l}\int_{-l/2}^{l/2}ds\,\mathbf{u}(\mathbf{x}_0+s\mathbf{p})\right)\\ +s\frac{12}{l^3}(\mathbf{I}-\mathbf{pp})\int_{-l/2}^{l/2}ds\,s\,\mathbf{u}(\mathbf{x}_0+s\mathbf{p})\end{array}\right]$$

which gives:

$$u_\parallel(s)-\frac{1}{l}\int_{\Omega_1}ds'\,u_\parallel(s')=-\frac{2}{\eta}f_\parallel(s)$$

Note that this equation can only be uniquely inverted if $l_1\neq l$, otherwise the mean is left undetermined. Applying the integral operator on the left yields:

$$\int_{\Omega_1}ds\,u_\parallel(s)=-\frac{2}{\eta}\frac{l}{l_2}\int_{\Omega_1}ds\,f_\parallel(s)$$

yielding the lovely expression

$$u_\parallel(s)=-\frac{2}{\eta}\left(f_\parallel(s)+\frac{1}{l_2}\int_{\Omega_1}ds'\,f_\parallel(s')\right)$$

We then have

$$\dot{\mathbf{x}}_0=\frac{1}{l}\int_{-l/2}^{l/2}ds\,\mathbf{u}(\mathbf{x}_0+s\mathbf{p})+\frac{2}{\eta}\frac{1}{l_2}\int_{\Omega_1}ds\,f_\parallel(s)\mathbf{p}$$

We also have

$$\mathbf{g}=\eta\left(\mathbf{I}-\frac{1}{2}\mathbf{pp}\right)\left[\begin{array}{c}-\left(\mathbf{u}(\mathbf{x}_0+s\mathbf{p})-\frac{1}{l}\int_{-l/2}^{l/2}ds'\,\mathbf{u}(\mathbf{x}_0+s'\mathbf{p})\right)\\ +s\frac{12}{l^3}(\mathbf{I}-\mathbf{pp})\int_{-l/2}^{l/2}ds'\,s'\,\mathbf{u}(\mathbf{x}_0+s'\mathbf{p})\end{array}\right]$$

Finally, for the remaining force:

$$\mathbf{f}_2=\eta\left[\begin{array}{c}\frac{1}{\eta}\frac{1}{l_2}\int_{\Omega_1}ds\,f_\parallel(s)\mathbf{p}\\ -\left(\mathbf{I}-\frac{1}{2}\mathbf{pp}\right)\left(\mathbf{u}(\mathbf{x}_0+s\mathbf{p})-\frac{1}{l}\int_{-l/2}^{l/2}ds\,\mathbf{u}(\mathbf{x}_0+s\mathbf{p})\right)\\ +s\frac{12}{l^3}(\mathbf{I}-\mathbf{pp})\int_{-l/2}^{l/2}ds\,s\,\mathbf{u}(\mathbf{x}_0+s\mathbf{p})\end{array}\right]$$

(iii) Now, let's assume that $l$ is small relative to the scale of the linear flow, so that $\mathbf{u}(\mathbf{x}_0+s\mathbf{p})\approx\mathbf{u}(\mathbf{x}_0)+s\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}$:

$$\dot{\mathbf{x}}_0=\mathbf{u}(\mathbf{x}_0)+\frac{2}{\eta}\frac{1}{l_2}\int_{\Omega_1}ds\,f_\parallel(s)\mathbf{p}$$

$$\dot{\mathbf{p}}=(\mathbf{I}-\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}$$

$$\mathbf{g}=-\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}s$$

$$\mathbf{f}_2=\frac{1}{l-l_1}\int_{\Omega_1}ds\,f_\parallel(s)\mathbf{p}-\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}s$$

$$\mathbf{f} = \chi_1(-f_\parallel(s)\mathbf{p} + \mathbf{g}) + \chi_2\,\mathbf{f}_2$$
$$= -\chi_1 f_\parallel(s)\mathbf{p} + \chi_2\frac{1}{l_2}\int_{\Omega_1} ds\, f_\parallel(s)\mathbf{p} - \frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}s$$

(iii.a) Single particle input power calculation

$$P(t) = \int ds\,\mathbf{f}^T\mathbf{u}$$
$$= \int ds\left[-\chi_1 f_\parallel(s)\mathbf{p} + \chi_2\frac{1}{l_2}\int_{\Omega_1} ds\, f_\parallel(s)\mathbf{p} - s\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}\right]$$
$$\cdot\left[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}} - \chi_1\frac{2}{\eta}\left(f_\parallel(s) + \frac{1}{l_2}\int_{\Omega_1} ds'\, f_\parallel(s')\right)\mathbf{p}\right]$$
$$= \int_{\Omega_1} ds\left[-f_\parallel(s)\mathbf{p} - s\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}\right]$$
$$\cdot\left[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}} - \frac{2}{\eta}\left(f_\parallel(s) + \frac{1}{l_2}\int_{\Omega_1} ds'\, f_\parallel(s')\right)\mathbf{p}\right]$$
$$+\int_{\Omega_2} ds\left[\frac{1}{l_2}\int_{\Omega_1} ds\, f_\parallel(s)\mathbf{p} - s\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}(\mathbf{x}_0)\mathbf{p}\right]\cdot[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}}]$$

Let’s assume that $f_\parallel = Const$, then

$$\dot{\mathbf{x}}_0 = \mathbf{u}_0 + \frac{2}{\eta}\frac{l_1}{l_2}f_\parallel\mathbf{p}$$

$$P(t) = \int_{\Omega_1} ds\left[-f_\parallel\mathbf{p} - s\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}_0\mathbf{p}\right]\cdot\left[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}} - \frac{2}{\eta}\left(1 + \frac{l_1}{l_2}\right)f_\parallel\mathbf{p}\right]$$
$$+\int_{\Omega_2} ds\left[\frac{l_1}{l_2}f_\parallel\mathbf{p} - s\frac{\eta}{2}(\mathbf{pp})\nabla\mathbf{u}_0\mathbf{p}\right]\cdot[\dot{\mathbf{x}}_0 + s\dot{\mathbf{p}}]$$
$$= -f_\parallel l_1\left[\mathbf{p}\cdot\mathbf{u}_0 - \frac{2}{\eta}f_\parallel\right] - \frac{\eta}{2}\left(\int_{\Omega_1} ds\, s\right)\left[\mathbf{u}_0 - \frac{2}{\eta}f_\parallel\mathbf{p}\right]^T[\mathbf{pp}^T\nabla\mathbf{u}_0\mathbf{p}]$$
$$+ f_\parallel l_1\left[\mathbf{p}\cdot\mathbf{u}_0 + \frac{2}{\eta}\frac{l_1}{l_2}f_\parallel\right] - \frac{\eta}{2}\left(\int_{\Omega_2} ds\, s\right)\left[\mathbf{u}_0 + \frac{2}{\eta}\frac{l_1}{l_2}f_\parallel\mathbf{p}\right]^T[\mathbf{pp}^T\nabla\mathbf{u}_0\mathbf{p}]$$
$$= \frac{2}{\eta}l\frac{l_1}{l_2}f_\parallel^2 + \frac{l_1}{l_2}f_\parallel\left(\int_{\Omega_1} ds\, s - \int_{\Omega_2} ds\, s\right)\mathbf{p}^T\nabla\mathbf{u}_0\mathbf{p}$$

Ok, there it is, though it needs checking (don’t like this model giving infinity as $l_2 \to 0$. Note that the first term is independent of being either Pusher or Puller. The second is not, and changes sign accordingly:

**Pusher**: Let $\Omega_1 = [-l/2, 0]$, $\Omega_2 = [0, l/2]$. Then

$$P = \frac{2l}{\eta}f_\parallel^2 - f_\parallel\frac{l^2}{8}\mathbf{p}^T\nabla\mathbf{u}_0\mathbf{p}$$

**Puller**: Let $\Omega_1 = [0, l/2]$, $\Omega_2 = [-l/2, 0]$. Then

$$P = \frac{2l}{\eta}f_\parallel^2 + f_\parallel\frac{l^2}{8}\mathbf{p}^T\nabla\mathbf{u}_0\mathbf{p}$$

I think that there is a sign error somewhere along the line. Check. Still, the two cases give oppositely-signed contributions relative to the first term. This sets the baseline in the paper.

**Side calculation**: Power for single SS swimmer, using David’s notation from the supplementary material notes. Now swimmer is length $2l$.

$$P_{ss} = \int_{-l}^{+l} ds\, \mathbf{u} \cdot \mathbf{f}$$

with

$$\mathbf{f} = \left[ \alpha(s) f_0(s) m_1(s) r(s) - \frac{m(s)}{M} \int_{-l}^{+l} ds'\, \alpha(s') f_0(s') m_1(s') r(s') \right] \mathbf{p}$$

$$\mathbf{u} = [U + \alpha(s) m_1(s) u_s(s)]\mathbf{p}$$

and

$$U = -\frac{2c}{M} \int_{-l}^{+l} ds'\, \alpha(s') f_0(s') m_1(s') r(s')$$

$$u_s(s) = 2c f_0(s) r(s) - m_1(s) U$$

yielding

$$\begin{aligned}
P_{ss} &= \int_{-l}^{+l} ds\, \alpha(s) m_1(s) u_s(s) \left[ \begin{array}{c} \alpha(s) f_0(s) m_1(s) r(s) \\ -\frac{m(s)}{M} \int_{-l}^{+l} ds'\, \alpha(s') f_0(s') m_1(s') r(s') \end{array} \right] \\
&= \int_{-l}^{+l} ds\, \alpha(s) m_1(s) u_s(s) \left[ \alpha(s) f_0(s) m_1(s) r(s) + \frac{U}{2c} m(s) \right] \\
&= \int_{-l}^{+l} ds\, \alpha(s) m_1(s) [2c f_0(s) r(s) - m_1(s) U] \left[ \alpha(s) f_0(s) m_1(s) r(s) + \frac{U}{2c} m(s) \right] \\
&= \int_{-l}^{+l} ds\, \alpha(s) m_1(s) \left[ \begin{array}{c} 2c f_0^2(s) \alpha(s) m_1(s) r^2(s) \\ +U f_0(s) r(s) [1 - 2\alpha(s) m_1^2(s)] \\ -m_1(s) \frac{U^2}{2c} m(s) \end{array} \right] \\
&= -\frac{U^2 M}{2c} + \int_{-l}^{+l} ds \left[ \begin{array}{c} 2c f_0^2(s) \alpha^2(s) m_1^2(s) r^2(s) - \frac{U^2}{2c} \alpha(s) m_1^2(s) m(s) \\ -2U \alpha^2(s) m_1^3(s) f_0(s) r(s) \end{array} \right]
\end{aligned}$$

What a mess; Don't really see what to do...

(iii.b) Single particle extra stress calculation:

$$\begin{aligned}
\int ds\, \mathbf{f}\, \mathbf{x}^T &= \int_{-l/2}^{l/2} ds\, s \left[ \begin{array}{c} -\chi_1 f_\parallel(s) \mathbf{p} + \chi_2 \frac{1}{l - l_1} \int_{\Omega_1} ds\, f_\parallel(s) \mathbf{p} \\ -\frac{\eta}{2} (\mathbf{p}\mathbf{p}^T) \nabla \mathbf{u}(\mathbf{x}_0) \mathbf{p} s \end{array} \right] \mathbf{p}^T \\
&= \left[ \begin{array}{c} \left( \frac{1}{l_2} \int_{\Omega_1} ds\, f_\parallel(s) \cdot \int_{\Omega_2} ds\, s - \int_{\Omega_1} s f_\parallel(s) \right) \mathbf{p}\mathbf{p}^T \\ -\frac{\eta}{2} \frac{l^3}{12} (\mathbf{p}\mathbf{p}^T) \nabla \mathbf{u}(\mathbf{x}_0) \mathbf{p}\mathbf{p}^T \end{array} \right]
\end{aligned}$$

Simplify further by assuming that $f_\parallel = Const$, so that

$$\dot{\mathbf{x}}_0 = \mathbf{u}(\mathbf{x}_0) + \frac{2}{\eta} \frac{l_1}{l_2} f_\parallel\, \mathbf{p}$$

$$\int ds\, \mathbf{f}\, \mathbf{x}^T = \left[ f_\parallel \left( \frac{l_1}{l_2} \int_{\Omega_2} ds\, s - \int_{\Omega_1} ds\, s \right) - \frac{\eta}{2} \frac{l^3}{12} (\mathbf{p}\mathbf{p}^T) : \nabla \mathbf{u} \right] \mathbf{p}\mathbf{p}^T$$

This form is pretty interesting because it involves first moments of the regions where propulsive stress and no-slip are separately applied, and their difference determines whether one has a pusher or a puller. Also, shear-thinning etc should drop immediately out of this.

# b. The extra stress calculation

$$\langle \boldsymbol{\sigma} \rangle = -\frac{1}{l_b^3} \sum_m^M \int_{B_m} dS\, \mathbf{g}\mathbf{X}^T = -\frac{1}{l_b^3} \sum_m^M \int_{\Gamma_m} ds\, \mathbf{f}\mathbf{x}^T$$

$$= -\frac{N}{L^3} \frac{M/l_b^3}{N/L^3} (\kappa_1 l^3 g_\parallel) \frac{1}{M} \sum_m^M \mathbf{p}_m \mathbf{p}_m = -n\, S\, C\, \frac{1}{M} \sum_m^M \mathbf{p}_m \mathbf{p}_m$$

where $S = \kappa_1 l^3 g_\parallel$ (units of force x length) , and $C = (M/l_b^3)/(N/L^3)$ is the local concentration.

# c. Scaling

The normalization from SS2008 is

$$\frac{1}{L^3} \int_\Omega dV_x \int_S dS_p\, \Psi = n = \frac{N}{L^3}$$

Now rescale as $x \to l_c x$, $u \to Uu$, and $\Psi \to n\Psi$. Normalization becomes

$$\frac{1}{(L/l_c)^3} \int_\Omega dV_x \int_S dS_p\, \Psi = 1$$

where $\Psi = 1/4\pi$ if $\Psi$ is a constant. Fluxes become

$$\dot{\mathbf{x}} = \mathbf{p} + \mathbf{u} - \frac{D_p}{l_c U} \nabla_x \ln \Psi$$

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp}) \nabla_x \mathbf{u}\, \mathbf{p} - \frac{d_p l_c}{U} \nabla_p \ln \Psi$$

and momentum balance:

$$-\Delta \mathbf{u} + \nabla \mathbf{q} = \frac{l_c^2}{\mu U} \frac{1}{l_c} n S\, \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

$$= l_c \frac{n \kappa_1 l^3 g_\parallel}{\mu \kappa_2 l g_\parallel / \mu} \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

$$= l_c \frac{N l^3}{L^3} l^{-1}\, \frac{\kappa_1}{\kappa_2} \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

$$= l_c (\nu l^{-1}) \frac{\kappa_1}{\kappa_2} \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

So, choose $l_c = l/\nu$ and $\alpha = \kappa_1/\kappa_2$ so that

$$-\Delta \mathbf{u} + \nabla \mathbf{q} = \alpha\, \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

and the fluxes are then

$$\dot{\mathbf{x}} = \mathbf{p} + \mathbf{u} - \left( \frac{D_p \nu}{l U} \right) \nabla_x \ln \Psi$$

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp}) \nabla_x \mathbf{u}\, \mathbf{p} - \left( \frac{d_p l}{\nu U} \right) \nabla_p \ln \Psi$$

If as observed at low $\nu$ in SS2007 that $d_p = \nu \bar{d}_p$ and $D_p = \bar{D}_p/\nu$, we have

$$\dot{\mathbf{x}} = \mathbf{p} + \mathbf{u} - \left( \frac{\bar{D}_p}{lU} \right) \nabla_x \ln \Psi$$

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp}) \nabla_x \mathbf{u} \, \mathbf{p} - \left( \frac{\bar{d}_p l}{U} \right) \nabla_p \ln \Psi$$