Global Stability of 2D Plane Couette Flow beyond the Energy Stability Limit

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1 Introduction

The field of hydrodynamic stability studies the transient responses of an initial perturbation around a known steady flow. Due to being fundamental in the understanding of transition to turbulence, it has garnered the attention of influential scientists over the years, including Reynolds, Orr, and Heisenberg, among others [4].

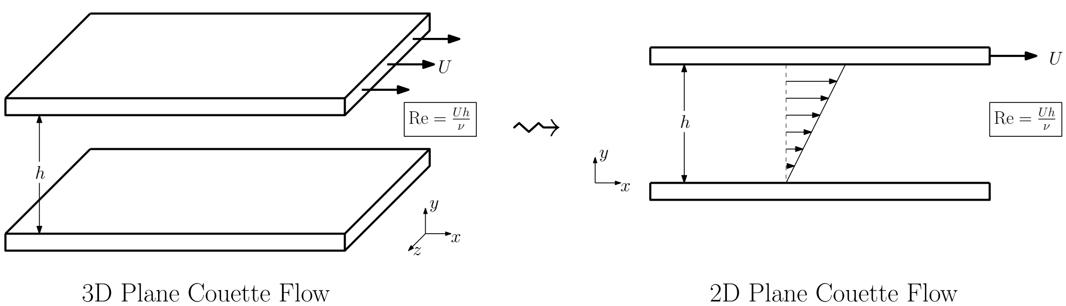

Figure 1.1: Diagram illustrating plane Couette flow (in 3D and 2D) with the corresponding system of coordinates, Reynolds number and steady laminar flow.

Sound theoretical results matching experiments have readily been found in many cases, such as Taylor-Couette flow [17], but others have remained more elusive. One classical example is plane Couette flow, which, as shown in Figure 1.1, is the flow between two infinite parallel plates separated by a distance h with the top plate moving at speed U in a direction parallel to the plates. The Reynolds number is defined as $\mathrm{Re} = \frac{Uh}{\nu} > 0$, where ν is the kinematic viscosity, while x, y and z are called the axial, wall-normal and transverse directions respectively. A linear shear flow is well-known to be the steady laminar equilibrium (see Figure 1.1), and it is the evolution of perturbations about this flow that are of interest.

This flow was proved by Romanov [15] to have a linear stability limit of $\mathrm{Re}_L = \infty$, meaning that it is *linearly* stable for *any* Reynolds number. That is, there always exist initial perturbations, $\mathbf{u}(0)$, under an infinitesimal energy which will decay (i.e., $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$) to the laminar steady flow. Hence, other notions of stability must be studied to explain when and how the flow becomes unstable. With this in mind, it is important to look at *global* stability, meaning that *every* initial perturbation $\mathbf{u}(0)$ decays, and *conditional* stability, meaning that a certain subset of initial perturbations $\mathbf{u}(0)$ decay. The global stability limit, denoted Re_G, is the largest number under which the flow is globally stable.

The problem of finding the global stability limit of 2D plane Couette flow dates back to 1907 [13], over 110 years ago, when William McFadden Orr found a lower bound for this flow's global stability limit, the so-called energy stability limit, $\mathrm{Re}_E = 177.2 \leq \mathrm{Re}_G$. He did so using what we now call the *energy method*, first proposed by Reynolds shortly before, and since then, this has been the only systematic mechanism to rigorously establish lower bounds to the global stability limit of fluid flows. In some rare situations, ad hoc techniques can be developed to find a larger lower bound, but in general this is a difficult task, which, in over a century, has proved to be an unsuccessful endeavor for the particular case of plane Couette flow. The purpose of this work is to find a better lower bound to the Re_G of 2D plane Couette flow using techniques that can additionally be applied to other flows in the future.

It is worth noting that 2D plane Couette flow (right of Figure 1.1) is the transversely-independent ($\frac{\partial(\cdot)}{\partial z} = 0$) simplification of 3D plane Couette flow (left of Figure 1.1). In 3D plane Couette flow, the energy stability limit was proved to be actually lower than the 2D counterpart, $\mathrm{Re}_E = 82.6$ [1, 11, 10, 16], but once again no lower bound to Re_G beyond Re_E has been established. A theoretical upper bound to Re_G was determined for 3D plane Couette flow by the 3D finite-amplitude periodic solutions found by Nagata [12], which occur at about $\mathrm{Re} = 500$. Meanwhile, experiments place upper bounds of Re_G on 3D plane Couette flow to be around $\mathrm{Re} = 1300$ [8, 3, 18]. By contrast, no upper bounds of Re_G have been found for 2D plane Couette flow either through theoretical means or numerical simulations [14]. Thus, it could be true that $\mathrm{Re}_G = \infty$ for 2D plane Couette flow, and finding a set of increasing lower bounds of Re_G could shed some light onto this open question.

This report is organized as follows. In Section 2 a brief review of how to write a fluid system as an uncertain dynamical system will be given. The energy method used to prove global stability will be described, and an alternative based on new techniques coming from sum-of-squares (SOS) polynomials optimization will be introduced. These techniques allow to reduce the problem to a tractable semidefinite program (SDP) which can be solved using a computer. They will produce high-order Lyapunov functions more general than the energy. In Section 3, solving the energy eigenvalue problem for 2D plane Couette flow will be outlined. Section 4 will present the results and discussion, while Section 5 will contain the concluding remarks. Lastly, the family of Appendices A–E will have extensive technical details associated to the computations and mathematical derivations.

2 Review of Fluid Dynamical Systems

What follows is a brief review of the uncertain fluid dynamical system first presented in [7].

Assume Ω is a bounded domain, and boundary conditions for the fluid velocity $\mathbf{v}$ and pressure p_0 consist of a combination of fixed known velocities and periodicity of the velocity and pressure fields, which additionally satisfy the nondimensional Navier-Stokes equations,

$$\begin{aligned} \tfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p_0 + \tfrac{1}{\mathrm{Re}} \nabla^2 \mathbf{v} + \mathbf{f}_g \,, \\ \nabla \cdot \mathbf{v} &= 0 \,, \end{aligned} \tag{2.1}$$

where $\mathbf{f}_g$ represents the gravity effects. Provided a steady solution, $\mathbf{V}$ and P, is known, the Navier-Stokes equations become

$$\begin{aligned} \tfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{u} &= -\nabla p + \tfrac{1}{\mathrm{Re}}\nabla^2\mathbf{u}\,, \\ \nabla\cdot\mathbf{u} &= 0\,, \end{aligned} \tag{2.2}$$

where the unknown perturbations $\mathbf{u} = \mathbf{v} - \mathbf{V}$ and $p = p_0 - P$ satisfy no-slip boundary conditions ($\mathbf{u} = \mathbf{0}$) wherever $\mathbf{v}$ is fixed, and periodic boundary conditions elsewhere.

Consider the following series expansion for the perturbation velocity field,

$$\begin{gathered} \mathbf{u}(\mathbf{x},t) = \sum_{i=1}^{m} a_i(t)\mathbf{u}_i(\mathbf{x}) + \mathbf{u}_\mathbf{s}(\mathbf{x},t)\,, \\ \nabla\cdot\mathbf{u}_i = 0\,, \qquad \langle \mathbf{u}_i\,,\mathbf{u}_j\rangle = \delta_{ij}\,, \qquad \nabla\cdot\mathbf{u}_\mathbf{s} = 0\,, \qquad \langle \mathbf{u}_\mathbf{s}\,,\mathbf{u}_i\rangle = 0\,, \end{gathered} \tag{2.3}$$

where δ_{ij} is the Kronecker delta. Hence, the basis fields, $\mathbf{u}_i$, and the residual perturbation velocity, $\mathbf{u}_\mathbf{s}$, are solenoidal, meaning their divergence vanishes and implying that the incompressibility of the perturbation velocity is satisfied, $\nabla\cdot\mathbf{u} = 0$. Moreover, the $\mathbf{u}_i$ are orthonormal in the L^2 inner product, and $\mathbf{u}_\mathbf{s}$ is orthogonal to all the $\mathbf{u}_i$. Here, $\langle\cdot\,,\cdot\rangle$ is the L^2 inner product, so that

$$\langle \mathbf{w}_1\,,\mathbf{w}_2\rangle = \int_\Omega \mathbf{w}_1\cdot\mathbf{w}_2\,\mathrm{d}\Omega\,, \qquad \|\mathbf{w}\|^2 = \langle\mathbf{w}\,,\mathbf{w}\rangle = \int_\Omega |\mathbf{w}|^2\,\mathrm{d}\Omega\,, \tag{2.4}$$

where $\|\cdot\|$ is the L^2 norm, and $|\cdot|$ is the usual Euclidean norm of a vector.

Next, let $\mathbf{a}(t) = [a_1(t),\ldots,a_m(t)] \in \mathbb{R}^m$ and $q(t) = \|\mathbf{u}_\mathbf{s}(t)\|$, so that the perturbation energy is $\|\mathbf{u}(t)\|^2 = |\mathbf{a}(t)|^2 + q^2(t)$. Chernyshenko and Goulart [7] show that the dynamical system $\widetilde{\mathbf{a}}(t) = (\mathbf{a}(t), q^2(t))$ describing the perturbation velocity is

$$\begin{aligned} \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} &= \mathbf{f}(\mathbf{a}) + \boldsymbol{\Theta}_\mathbf{a}(\mathbf{u}_\mathbf{s}) + \boldsymbol{\Theta}_\mathbf{b}(\mathbf{u}_\mathbf{s},\mathbf{a}) + \boldsymbol{\Theta}_\mathbf{c}(\mathbf{u}_\mathbf{s})\,, \\ \frac{1}{2}\frac{\mathrm{d}q^2}{\mathrm{d}t} &= -\mathbf{a}\cdot\Big(\boldsymbol{\Theta}_\mathbf{a}(\mathbf{u}_\mathbf{s}) + \boldsymbol{\Theta}_\mathbf{b}(\mathbf{u}_\mathbf{s},\mathbf{a}) + \boldsymbol{\Theta}_\mathbf{c}(\mathbf{u}_\mathbf{s})\Big) + \Gamma(\mathbf{u}_\mathbf{s}) + \chi(\mathbf{u}_\mathbf{s},\mathbf{a})\,, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} f_i(\mathbf{a}) &= \overbrace{\Big(\tfrac{1}{\mathrm{Re}}\langle\mathbf{u}_i\,,\nabla^2\mathbf{u}_j\rangle - \langle\mathbf{u}_i\,,\mathbf{u}_j\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{u}_j\rangle\Big)a_j}^{(\mathbf{L}\cdot\mathbf{a})_i} + \overbrace{\big(-\langle\mathbf{u}_i\,,\mathbf{u}_j\cdot\nabla\mathbf{u}_k\rangle\big)a_j a_k}^{(\boldsymbol{\mathcal{N}}:\mathbf{a}\otimes\mathbf{a})_i}\,, \\ \Theta_{\mathbf{a}i}(\mathbf{u}_\mathbf{s}) &= \langle\mathbf{u}_\mathbf{s}\,,\mathbf{h}_{i0}\rangle\,, \qquad \mathbf{h}_{i0} = \tfrac{1}{\mathrm{Re}}\nabla^2\mathbf{u}_i + \mathbf{V}\cdot\nabla\mathbf{u}_i - \mathbf{u}_i\cdot\nabla^\mathsf{T}\mathbf{V}\,, \\ \Theta_{\mathbf{b}i}(\mathbf{u}_\mathbf{s},\mathbf{a}) &= \langle\mathbf{u}_\mathbf{s}\,,\mathbf{h}_{ij}\rangle a_j\,, \qquad \mathbf{h}_{ij} = \mathbf{u}_j\cdot\nabla\mathbf{u}_i - \mathbf{u}_i\cdot\nabla^\mathsf{T}\mathbf{u}_j\,, \\ \Theta_{\mathbf{c}i}(\mathbf{u}_\mathbf{s}) &= \langle\mathbf{u}_\mathbf{s}\,,\mathbf{u}_\mathbf{s}\cdot\nabla\mathbf{u}_i\rangle\,, \\ \Gamma(\mathbf{u}_\mathbf{s}) &= \tfrac{1}{\mathrm{Re}}\langle\mathbf{u}_\mathbf{s}\,,\nabla^2\mathbf{u}_\mathbf{s}\rangle - \langle\mathbf{u}_\mathbf{s}\,,\mathbf{D}\mathbf{u}_\mathbf{s}\rangle\,, \\ \chi(\mathbf{u}_\mathbf{s},\mathbf{a}) &= 2\langle\mathbf{u}_\mathbf{s}\,,\mathbf{d}_j\rangle a_j\,, \qquad \mathbf{d}_j = \tfrac{1}{\mathrm{Re}}\nabla^2\mathbf{u}_j - \mathbf{D}\mathbf{u}_j\,, \end{aligned} \tag{2.6}$$

with $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{V} + \nabla^\mathsf{T}\mathbf{V})$ being the rate of strain tensor of the steady flow $\mathbf{V}$, and where $\nabla^\mathsf{T}\mathbf{w} = (\nabla\mathbf{w})^\mathsf{T}$ for any vector field $\mathbf{w}$ (e.g. $(\mathbf{h}_{ij})_k = (\mathbf{u}_j)_l(\nabla\mathbf{u}_i)_{lk} - (\mathbf{u}_i)_l(\nabla\mathbf{u}_j)_{kl}$ for $j \geq 1$, where $(\nabla\mathbf{w})_{ij} = \frac{\partial w_i}{\partial x_j}$).

The evolution of the fluid dynamical system described through $\frac{\mathrm{d}\widetilde{\mathbf{a}}}{\mathrm{d}t}$ in (2.5), is nonlinear in $\mathbf{a}$ (via $\boldsymbol{\mathcal{N}} : \mathbf{a} \otimes \mathbf{a}$ in $\boldsymbol{\xi}(\mathbf{a})$), and more importantly it is *uncertain* in q due to the fact that it is multivalued in that variable. Indeed, for a fixed q, there are multiple values $\mathbf{u_s}$ such that $q = \|\mathbf{u_s}\|$ meaning that $\frac{\mathrm{d}\widetilde{\mathbf{a}}}{\mathrm{d}t}$ can take multiple values for a single value of q.

2.1 Lyapunov functionals and the energy method

Now, consider a real-valued Lyapunov functional $V(\mathbf{u})$, with $V(\mathbf{0}) = 0$. Then, Lyapunov's theorem says that if V is positive-definite and radially unbounded (i.e. $V(\mathbf{u}) > 0$ for all $\mathbf{u} \neq \mathbf{0}$ and $V(\mathbf{u}) \to \infty$ as $\|\mathbf{u}\| \to \infty$), and $\frac{\mathrm{d}V(\mathbf{u}(t))}{\mathrm{d}t}$ is negative definite (i.e. $\frac{\mathrm{d}V(\mathbf{u}(t))}{\mathrm{d}t} < 0$ for all $\mathbf{u} \neq \mathbf{0}$), it follows that the flow is globally asymptotically stable (meaning that for every initial perturbation $\mathbf{u}(0)$ it follows $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$).

The classical approach to proving global stability is to use the *energy method*, where the Lyapunov function is chosen as the perturbation energy, $V(\mathbf{u}) = E = \frac{1}{2}\|\mathbf{u}\|^2$. In the simplest case, where $m = 0$ in (2.3), so $\mathbf{u} = \mathbf{u_s}$ and $\widetilde{\mathbf{a}}(t) = q(t) = \|\mathbf{u}(t)\|$, then $V(\mathbf{u}(t)) = E(t) = \frac{1}{2}q^2(t)$, $\mathbf{a}$ does not exist and (2.5) becomes

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}q^2}{\mathrm{d}t} = \Gamma(\mathbf{u}) = \frac{1}{\mathrm{Re}}\langle \mathbf{u}\,, \nabla^2\mathbf{u}\rangle - \langle \mathbf{u}\,, \mathbf{D}\mathbf{u}\rangle\,. \tag{2.7}$$

It is clear $V(\mathbf{0}) = 0$, $V(\mathbf{u}) > 0$ for all $\mathbf{u} \neq \mathbf{0}$ and more importantly $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t}$. Therefore, if $\frac{\mathrm{d}V}{\mathrm{d}t} \leq \kappa_\mathbf{s} q^2$ for some $\kappa_\mathbf{s} < 0$, the flow will be globally stable.

Thus, solving for the minimum $\kappa_\mathbf{s} \in \mathbb{R}$ such that $\frac{\mathrm{d}V}{\mathrm{d}t} \leq \kappa_\mathbf{s} q^2$ yields a constrained minimization problem which is equivalent to an eigenvalue problem known as the *energy eigenvalue problem*,

$$\begin{aligned} -\lambda\mathbf{u} &= \mathbf{D}\mathbf{u} - \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} + \nabla\zeta\,, \\ \nabla \cdot \mathbf{u} &= 0\,, \end{aligned} \tag{2.8}$$

which is solved for $\mathbf{u}$ and ζ satisfying the same boundary conditions as the perturbation velocity and pressure in (2.2). Its solution is the eigenvalues and eigenfunctions of the Hermitian (symmetric) *energy operator*, $\mathbf{A}_E\mathbf{u} = \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} - \mathbf{D}\mathbf{u} - \nabla\zeta$, where ζ depends on $\mathbf{u}$ through the auxiliary Poisson problem $\nabla^2\zeta = -\nabla \cdot \mathbf{D}\mathbf{u}$ with the boundary conditions $\nabla\zeta \cdot \mathbf{n} = (\frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} - \mathbf{D}\mathbf{u}) \cdot \mathbf{n}$ wherever $\mathbf{u}$ has no-slip boundary conditions, and periodic otherwise. Recalling the meaning of $\kappa_\mathbf{s}$, yields that the flow is globally stable provided the largest eigenvalue of the energy operator is negative. The *energy stability limit*, Re_E, is obtained by solving for Re in limiting case in which the largest eigenvalue is 0. This implies any $\mathrm{Re} < \mathrm{Re}_E$ is associated to a negative eigenvalue, so that the flow is globally stable, and thus Re_E is a lower bound of the global stability limit, $\mathrm{Re}_G \geq \mathrm{Re}_E$.

Despite being very practical, it is clear that the energy method is simply a special choice of *quadratic* Lyapunov functional in the context of a much more general theorem. The ideal scenario would be to find other Lyapunov functionals that hopefully allow to establish that the flow can be globally stable for values of Re above Re_E. It is possible to deduce that if this is desired, then high-order (above quadratic) Lyapunov functionals that are not powers of the energy should be considered. In view of the form of the uncertain system (2.5), look at high-order polynomial functions of the form $V(\mathbf{u}) = V(\mathbf{a}, q^2)$ (with $m > 0$). Additionally, note that the uncertain terms in (2.5) do not impede the use of the Lyapunov theorem as long as the terms dependent on $\mathbf{u_s}$ (namely $\mathbf{\Theta_a}$, $\mathbf{\Theta_b}$, $\mathbf{\Theta_c}$, Γ and χ) are bounded in some sense in terms of a single valued function of $\mathbf{a}$ and q.

Before proceeding, it is useful to simplify (2.5) further if possible. In this sense, the energy eigenvalue problem is of practical use. Indeed, from now on, assume that the basis fields $\mathbf{u}_i$ for $i = 1, \ldots, m$ are a subset of the energy eigenfunctions. That is, for each $i = 1, \ldots, m$ assume there exist ζ_i and $\lambda_i \in \mathbb{R}$ such that $(\mathbf{u}_i, \zeta_i, \lambda_i)$ is a solution to the energy eigenvalue problem in (2.8). Under this assumption of the $\mathbf{u}_i$, it follows $\chi(\mathbf{u_s}, \mathbf{a}) = 0$ and $\Gamma(\mathbf{u_s}) \leq \kappa q^2$, where κ is the largest eigenvalue of (2.8) different from all the λ_i for $i = 1, \ldots, m$.

Then, provided $\frac{\partial V}{\partial q^2} \geq 0$, it is obvious that

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \underbrace{\frac{\partial V}{\partial \mathbf{a}} \cdot \mathbf{f}(\mathbf{a}) + 2\frac{\partial V}{\partial q^2}\Gamma(\mathbf{u_s})}_{G(\mathbf{a},q^2,\mathbf{u_s})} + \underbrace{\left(\frac{\partial V}{\partial \mathbf{a}} - 2\frac{\partial V}{\partial q^2}\mathbf{a}\right)}_{\mathbf{M}(\mathbf{a},q^2)} \cdot \big(\underbrace{\mathbf{\Theta_a}(\mathbf{u_s}) + \mathbf{\Theta_b}(\mathbf{u_s},\mathbf{a})}_{\mathbf{\Theta_{ab}}(\mathbf{u_s},\mathbf{a})} + \mathbf{\Theta_c}(\mathbf{u_s})\big) \\ &\leq \underbrace{\frac{\partial V}{\partial \mathbf{a}} \cdot \mathbf{f}(\mathbf{a}) + 2\frac{\partial V}{\partial q^2}\kappa q^2}_{\tilde{G}(\mathbf{a},q^2)} + \Xi(\mathbf{a}, q^2)\,, \end{aligned} \tag{2.9}$$

for some $\Xi(\mathbf{a}, q^2)$ satisfying $\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\mathbf{\Theta_{ab}}(\mathbf{u_s}, \mathbf{a}) + \mathbf{\Theta_c}(\mathbf{u_s})\big) \leq \Xi(\mathbf{a}, q^2)$, with $\Xi(\mathbf{0}, 0) = 0$. Finally, the idea is to use sum-of-squares (SOS) polynomial constraints to setup a semi-definite program (SDP) that ensures that $V(\mathbf{0}, 0) = 0$ and $\Xi(\mathbf{0}, 0) = 0$, and that $V > 0$, $\frac{\partial V}{\partial q^2} \geq 0$ and $\tilde{G} + \Xi < 0$ whenever $(\mathbf{a}, q^2) \neq (\mathbf{0}, 0)$.

The details to construct a valid $\Xi(\mathbf{a}, q^2)$ and to properly set up the SDP are of tantamount importance (there is no unique way of doing this). Having said that, the details are quite technical and better left for the Appendices, where several techniques were attempted. The only details worth repeating here are the bounds of each component of $\mathbf{\Theta_{ab}}$ and $\mathbf{\Theta_c}$, first derived in [7, 9]. Let $\tilde{\mathbf{a}} = [1, a_1, \ldots, a_m]$, indexed from 0 so that $\tilde{a}_0 = 1$ and $\tilde{a}_i = a_i$ for $i = 1, \ldots, m$. Then, $\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a}) = \langle \mathbf{u_s}\,, \mathbf{h}_{ij} \rangle \tilde{a}_j$, and the bounds are,

$$\begin{aligned} |\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})| &\leq \sqrt{\tilde{\mathbf{a}}^\mathsf{T} \mathbf{R}_i \tilde{\mathbf{a}} q^2}\,, & (R_i)_{kl} &= \langle \widetilde{\mathbf{h}}_{ik}\,, \widetilde{\mathbf{h}}_{il} \rangle\,, \\ |\Theta_{\mathbf{c}i}(\mathbf{u_s})| &\leq C_i q^2\,, & C_i &= \|\rho(\mathbf{D}_i)\|_\infty = \sup_{\mathbf{x}\in\Omega} \rho\big(\mathbf{D}_i(\mathbf{x})\big)\,, \end{aligned} \tag{2.10}$$

where $\mathbf{D}_i = \frac{1}{2}(\nabla \mathbf{u}_i + \nabla^\mathsf{T} \mathbf{u}_i)$, and $\rho(\mathbf{D}_i(\mathbf{x}))$ is the spectral radius of $\mathbf{D}_i(\mathbf{x})$. Here, $\widetilde{\mathbf{h}}_{ij}$ is the solenoidal projection (so $\nabla \cdot \widetilde{\mathbf{h}}_{ij} = 0$) of $\mathbf{h}_{ij}$ such that $\langle \widetilde{\mathbf{h}}_{ij}\,, \mathbf{u}_k \rangle = 0$ for all $k = 1, \ldots, m$ and satisfying that $\widetilde{\mathbf{h}}_{ij} \cdot \mathbf{n} = 0$ ($\mathbf{n}$ is the outer normal) wherever the perturbation velocity has no-slip boundary conditions. These bounds are important as they eliminate $\mathbf{u_s}$ and yield expressions only in terms of $a_1, \ldots, a_m$ and q^2.

3 Solving for the Energy Eigenfunctions and Bounds

Recall the setup and coordinates of 2D plane Couette flow in Figure 1.1 (or Figure 3.1). Assuming all parameters are naturally nondimensionalised, note that the well-known steady solution is $\mathbf{V} = \left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right]$, which satisfies the nondimensional boundary conditions $\mathbf{V} = \mathbf{0}$ at $y = 0$ and $\mathbf{V} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ at $y = 1$.

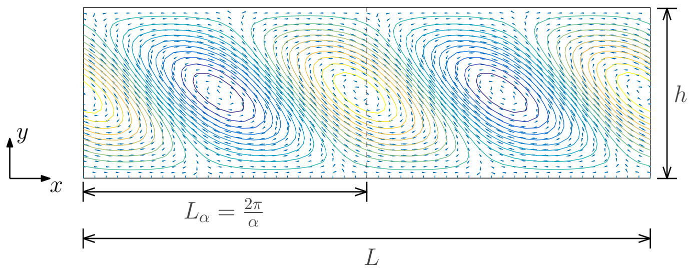

Figure 3.1: Diagram illustrating a periodic 2D plane Couette flow domain with the subperiods.

Assume a periodic 2D domain $\Omega = (0, L_x) \times (0, 1)$ as in Figure (3.1), and consider a perturbed velocity field $\mathbf{v} = \mathbf{V} + \mathbf{u}$ and pressure $p_0 = P + p$ with respect to the steady solution $\mathbf{V}$ and P, such that the perturbation velocity and pressure are $\mathbf{u} = \left[\begin{smallmatrix} u_x \\ u_y \end{smallmatrix}\right]$ and p and subject to no-slip boundary conditions at both plates $\mathbf{u}(x, 0) = \mathbf{u}(x, 1) = \mathbf{0}$.

The energy eigenvalue problem is written in (2.8), and from what is known from the flow, it follows that it can be simplified to

$$-\lambda \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} - \frac{1}{\mathrm{Re}} \begin{bmatrix} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \\ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \end{bmatrix} + \begin{bmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \end{bmatrix} \tag{3.1}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \tag{3.2}$$

To eliminate ζ one can take the (2D) curl of (3.1) to get

$$-\lambda \omega_z = \big(\nabla \times (\mathbf{D}\mathbf{u})\big)_z - \frac{1}{\mathrm{Re}} \nabla^2 \omega_z\,, \tag{3.3}$$

where $\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$ and $\big(\nabla \times (\mathbf{D}\mathbf{u})\big)_z = \frac{1}{2}\big(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\big)$

Moreover, since this is a 2D problem, there must exist a stream function ψ automatically satisfying the continuity equation (3.2) with

$$u_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\partial \psi}{\partial x}. \tag{3.4}$$

In this case, verification of $\omega_z = -\nabla^2 \psi$ is trivial, while $(\nabla \times (\mathbf{D}\mathbf{u}))_z = \frac{\partial^2 \psi}{\partial x \partial y}$, so (3.3) may be rewritten in terms of ψ as

$$\lambda \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x \partial y} + \frac{1}{\mathrm{Re}} \nabla^2 (\nabla^2 \psi). \tag{3.5}$$

The equation above is important, because it is sufficient (along with the boundary conditions) to solve the energy problem. The no-slip boundary conditions of the perturbation velocity in terms of the stream function are

$$u_x = \frac{\partial \psi}{\partial y} = 0 \quad \text{at} \quad y = 0 \text{ and } y = 1 \text{ for all } x \in (0, L_x), \tag{3.6a}$$

$$u_y = -\frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad y = 0 \text{ and } y = 1 \text{ for all } x \in (0, L_x). \tag{3.6b}$$

As mentioned before, the perturbation velocity is assumed periodic in the x direction, so ψ must accept a Fourier series expansion. It is written as

$$\psi(x,y) = \sum_{n\in\mathbb{Z}} \widehat{\psi}_n(y) e^{\mathrm{i}\alpha_n x}, \quad \text{where} \quad \alpha_n = \frac{2\pi}{L_x} n. \tag{3.7}$$

Since $\psi : \Omega \to \mathbb{R}$ is a real function, it easily follows that $\widehat{\psi}_{-n}(y) = \overline{\widehat{\psi}_n(y)}$, where the bar denotes complex conjugation. Substitution of (3.7) into equation (3.5) and the boundary conditions in (3.6) yield

$$\sum_{n\in\mathbb{Z}} \lambda\Big(-\alpha_n^2\widehat{\psi}_n + \tfrac{\mathrm{d}^2\widehat{\psi}_n}{\mathrm{d}y^2}\Big)e^{\mathrm{i}\alpha_n x} = \textstyle\sum_{n\in\mathbb{Z}}\Big(\mathrm{i}\alpha_n \tfrac{\mathrm{d}\widehat{\psi}_n}{\mathrm{d}y} + \tfrac{1}{\mathrm{Re}}\Big(\alpha_n^4\widehat{\psi}_n - 2\alpha_n^2\tfrac{\mathrm{d}^2\widehat{\psi}_n}{\mathrm{d}y^2} + \tfrac{\mathrm{d}^4\widehat{\psi}_n}{\mathrm{d}y^4}\Big)\Big)e^{\mathrm{i}\alpha_n x}, \tag{3.8}$$

$$\tfrac{\partial\psi}{\partial y} = \textstyle\sum_{n\in\mathbb{Z}} \tfrac{\mathrm{d}\widehat{\psi}_n}{\mathrm{d}y} e^{\mathrm{i}\alpha_n x} = 0 \quad \text{at} \quad y=0 \text{ and } y=1 \text{ for all } x\in(0,L_x), \tag{3.9a}$$

$$-\tfrac{\partial\psi}{\partial x} = -\textstyle\sum_{n\in\mathbb{Z}} \mathrm{i}\alpha_n \widehat{\psi}_n e^{\mathrm{i}\alpha_n x} = 0 \quad \text{at} \quad y=0 \text{ and } y=1 \text{ for all } x\in(0,L_x). \tag{3.9b}$$

Now, since the Fourier modes are known to be orthogonal with the $L^2(0,L_x)$ inner product, it follows that each Fourier mode can be treated separately. Therefore, dropping the subindex n, the equations above become

$$\lambda\Big(-\alpha^2\widehat{\psi} + \frac{\mathrm{d}^2\widehat{\psi}}{\mathrm{d}y^2}\Big) = \Big(\mathrm{i}\alpha\frac{\mathrm{d}\widehat{\psi}}{\mathrm{d}y} + \frac{1}{\mathrm{Re}}\Big(\alpha^4\widehat{\psi} - 2\alpha^2\frac{\mathrm{d}^2\widehat{\psi}}{\mathrm{d}y^2} + \frac{\mathrm{d}^4\widehat{\psi}}{\mathrm{d}y^4}\Big)\Big), \tag{3.10}$$

$$\widehat{\psi}(0) = \widehat{\psi}(1) = \frac{\mathrm{d}\widehat{\psi}}{\mathrm{d}y}(0) = \frac{\mathrm{d}\widehat{\psi}}{\mathrm{d}y}(1) = 0. \tag{3.11}$$

The equation is a fourth order homogeneous ordinary differential equation with constant coefficients and four vanishing boundary conditions. Hence, the solution is known to be a linear combination of exponentials. With this in mind, one first proposes that $\widehat{\psi}(y) = e^{\mathrm{i}\beta y}$. Substitution into (3.10) gives

$$-\lambda(\alpha^2+\beta^2) = -\alpha\beta + \tfrac{1}{\mathrm{Re}}(\alpha^2+\beta^2)^2\,, \tag{3.12}$$

which is the characteristic equation whose solutions are the roots of the characteristic polynomial

$$p_\psi(\lambda,\mathrm{Re},\alpha,\beta) = \tfrac{1}{\mathrm{Re}}(\alpha^2+\beta^2)^2 + \lambda(\alpha^2+\beta^2) - \alpha\beta\,, \tag{3.13}$$

with discriminant

$$\begin{aligned}\Delta_\psi(\lambda,\mathrm{Re},\alpha) = 256\alpha^8 &+ 384\alpha^6\lambda\mathrm{Re} - 27\alpha^4\mathrm{Re}^2 + 120\alpha^4\lambda^2\mathrm{Re}^2 \\ &+ 16\alpha^4\lambda^4\mathrm{Re}^2 - 4\alpha^2\lambda^3\mathrm{Re}^3 + 16\alpha^2\lambda^5\mathrm{Re}^3.\end{aligned} \tag{3.14}$$

The characteristic polynomial is symmetric in α and β. Given a fixed triplet $(\lambda,\mathrm{Re},\alpha)$, p_ψ will have exactly four roots $\beta_j(\lambda,\mathrm{Re},\alpha)$ for $j=1,\dots,4$, which are obviously dependent on that triplet. The roots β_j can even be computed analytically due to p_ψ being quartic in β. Moreover, if only real values of λ are considered, which is reasonable due to the operator being symmetric, then the coefficients in β of p_ψ will be real. This implies the roots β_j will either be real or come in conjugate pairs. Finally, since the coefficient of β^3 is 0 it follows that $\sum_{j=1}^{4} \beta_j = 0$.

For convenience, assume that all the roots are different, i.e. $\Delta_\psi(\lambda, \mathrm{Re}, \alpha) \neq 0$, so that due to linearity, the general solution of (3.10) will be

$$\widehat{\psi}(y) = C_1 e^{\mathrm{i}\beta_1 y} + C_2 e^{\mathrm{i}\beta_2 y} + C_3 e^{\mathrm{i}\beta_3 y} + C_4 e^{\mathrm{i}\beta_4 y}, \tag{3.15}$$

where the C_j are constant coefficients in $\mathbb{C}$.

The constant coefficients C_j in (3.15) are chosen to satisfy the boundary conditions (3.11). When $\Delta_\psi(\lambda, \mathrm{Re}, \alpha) \neq 0$, substituting (3.15) into (3.11) yields a linear system of equations, which in matrix form is

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{\mathrm{i}\beta_1} & e^{\mathrm{i}\beta_2} & e^{\mathrm{i}\beta_3} & e^{\mathrm{i}\beta_4} \\ \mathrm{i}\beta_1 & \mathrm{i}\beta_2 & \mathrm{i}\beta_3 & \mathrm{i}\beta_4 \\ \mathrm{i}\beta_1 e^{\mathrm{i}\beta_1} & \mathrm{i}\beta_2 e^{\mathrm{i}\beta_2} & \mathrm{i}\beta_3 e^{\mathrm{i}\beta_3} & \mathrm{i}\beta_4 e^{\mathrm{i}\beta_4} \end{bmatrix}}_{\mathbf{M}_\psi(\lambda,\mathrm{Re},\alpha)} \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}}_{\mathbf{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.16}$$

The complex matrix $\mathbf{M}_\psi$ is dependent on the triplet $(\lambda, \mathrm{Re}, \alpha)$ via the distinct roots β_j of $p_\psi(\lambda, \mathrm{Re}, \alpha, \beta)$. To have a nonzero eigenfunction it is then necessary for $\det(\mathbf{M}_\psi) = 0$, and $\mathbf{C} \in \ker(\mathbf{M}_\psi) \setminus \mathbf{0}$, which can then be substituted into (3.15) to calculate the complex function $\widehat{\psi}$. By adding the complementary Fourier mode, a real stream function is computed pointwise as

$$\psi(x, y) = \widehat{\psi}(y) e^{\mathrm{i}\alpha x} + \overline{\widehat{\psi}(y)} e^{-\mathrm{i}\alpha x} \in \mathbb{R}. \tag{3.17}$$

Then, a real eigenvelocity field $[u_x(x, y), u_y(x, y)]^\mathsf{T} \in \mathbb{R}^2$ corresponding to that stream function is easily determined via (3.4). In fact, if $\alpha \neq 0$, and $\mathbf{C} \in \ker(\mathbf{M}_\psi) \setminus \mathbf{0}$, then $\mathrm{i}\mathbf{C}$ is another relevant solution (shift by $\frac{\pi}{2}$) which leads to a shifted and linearly independent eigenvelocity field associated to the same eigenvalue. All eigenfunctions can be normalized (to have $\|\mathbf{u}\| = 1$).

If $\Delta_\psi(\lambda, \mathrm{Re}, \alpha) = 0$, then the necessary and tedious modifications associated having repeated roots must be done. This is left for the reader to ponder. Additionally, if $\alpha = 0$, then ψ is only a function of y and solving (3.1) directly gives $\mathbf{u}_i = [\sin(2\pi k y), 0]^\mathsf{T}$ with (unique) eigenvalues $\lambda_i = -\frac{(2\pi k)^2}{\mathrm{Re}}$ for $k \in \mathbb{N}$.

Next, assume that the Galerkin basis vector fields $\mathbf{u}_i$ for $i = 1, \ldots, m$ are chosen as eigenfunctions of the energy eigenvalue problem. The idea is to calculate the matrices $\mathbf{R}_i$ in (2.10). Let $\mathbf{b} = [b_x, b_y]^\mathsf{T}$ be either $\mathbf{h}_{i0}$ or $\mathbf{h}_{ij}$ for some $i, j = 1, \ldots, m$. The first step is to find $\phi_\mathbf{b}$ such that $\mathbf{b} = \breve{\mathbf{b}} + \nabla\phi_\mathbf{b}$, with $\nabla \cdot \breve{\mathbf{b}} = 0$ and $\breve{\mathbf{b}} \cdot \mathbf{n} = 0$ wherever the perturbation velocity has no-slip boundary conditions. This is equivalent to solving the Poisson problem

$$\nabla^2 \phi_\mathbf{b} = \nabla \cdot \mathbf{b}, \tag{3.18}$$

subject to the boundary conditions

$$\nabla\phi_\mathbf{b} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \tag{3.19}$$

wherever the perturbation velocity has no-slip boundary conditions. In this case no-slip boundary conditions occur at the plates, and those are precisely the points $(x, y) \in \partial\Omega$ for which $y = 0$ and $y = 1$. The outward unit normal $\mathbf{n}$ corresponding to those points are $\mathbf{n} = [0, -1]^\mathsf{T}$ when $y = 0$ and $\mathbf{n} = [0, 1]^\mathsf{T}$ when $y = 1$.

Proceeding as with the eigenvalue problem, the Fourier series expansions of $\phi_\mathbf{b}$ and $\mathbf{b}$ are

$$\phi_\mathbf{b}(x, y) = \sum_{n\in\mathbb{Z}} \widehat{\phi}_n(y) e^{\mathrm{i}\alpha_n x} \quad \text{and} \quad \mathbf{b}(x, y) = \sum_{n\in\mathbb{Z}} \widehat{\mathbf{b}}_n(y) e^{\mathrm{i}\alpha_n x}. \tag{3.20}$$

One can then substitute these expansions into (3.18) and treat each Fourier mode separately due to their orthogonality with the $L^2(\Omega)$ inner product. Therefore, dropping the n, the Poisson problem eventually becomes the nonhomogeneous second order ordinary differential equation

$$\frac{\mathrm{d}^2\widehat{\phi}}{\mathrm{d}y^2} - \alpha^2\widehat{\phi} = \frac{\mathrm{d}\widehat{b}_y}{\mathrm{d}y} + \mathrm{i}\alpha\widehat{b}_x, \tag{3.21}$$

with the boundary conditions

$$\frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}y}(0) = \widehat{b}_y(0) \quad \text{and} \quad \frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}y}(1) = \widehat{b}_y(1), \tag{3.22}$$

where $\widehat{\mathbf{b}}(y) = [\widehat{b}_x(y), \widehat{b}_y(y)]^\mathsf{T}$. This equation must be solved for each Fourier mode separately. Due to the form of $\mathbf{h}_{i0}$, $\mathbf{h}_{ij}$ and the energy eigenfunctions (only having 1 or 2 Fourier modes), it follows that $\mathbf{b}$ will have at most 4 separate Fourier modes.

The details will be skipped, but as usual, one must find first a general homogeneous solution $\widehat{\phi}_h(y)$ such that $\frac{\mathrm{d}^2\widehat{\phi}_h}{\mathrm{d}y^2} - \alpha^2\widehat{\phi}_h = 0$, followed by a particular solution $\widehat{\phi}_p$. This can actually be done analytically for the current problem. In the end, the full solution will be $\widehat{\phi} = \widehat{\phi}_h + \widehat{\phi}_p$, and $\phi_\mathbf{b}$ can be reconstructed via (3.20).

This means $\breve{\mathbf{b}} = \mathbf{b} - \nabla\phi_\mathbf{b}$ is known explicitly, and then $\widetilde{\mathbf{b}}$ is easy to compute as

$$\widetilde{\mathbf{b}} = \mathbf{b} - \nabla\phi_\mathbf{b} - \sum_{j=1}^{m} \langle\breve{\mathbf{b}}, \mathbf{u}_j\rangle\mathbf{u}_j. \tag{3.23}$$

Finally, one can proceed to calculate the integrals to find $\mathbf{R}_i$ for each $i = 1, \ldots, m$.

As mentioned before, $\kappa_\mathbf{s}$ is the $m + 1$ largest eigenvalue, provided the Galerkin basis vector fields $\mathbf{u}_i$ for $i = 1, \ldots, m$ are chosen as eigenfunctions associated to the largest eigenvalues. Their strain rate tensor is $\mathbf{D}_i$ for all $i = 1, \ldots, m$. Now, given that the flow is two dimensional and using the incompressibility of the eigenfunctions in the form $\mathrm{tr}(\mathbf{D}_i(\mathbf{x})) = 0$ for some arbitrary $\mathbf{x} \in \Omega$, it follows that the eigenvalues of $\mathbf{D}_i(\mathbf{x})$ can be computed explicitly. They have the same magnitude, which must be the spectral radius. Hence, the spectral radius $\rho(\mathbf{D}_i) : \Omega \to \mathbb{R}$ as a scalar field is

$$\rho(\mathbf{D}_i(\mathbf{x})) = \sqrt{\left(\frac{\partial^2\psi_i}{\partial x\partial y}\right)^2 + \frac{1}{4}\left(\nabla^2\psi_i\right)^2}. \tag{3.24}$$

This immediately implies that the problem of finding $||\rho(\mathbf{D}_i(\mathbf{x}))||_\infty$ becomes much easier. Nevertheless, it is solved numerically by being formulated as a constrained optimization problem of finding the global maximum of $\rho(\mathbf{D}_i)$ constrained to $\mathbf{x} \in \Omega$.

4 Results

Solving for the energy eigenmodes and bounds for 2D plane Couette flow as described in Section 3 allows to setup an optimization problem with sum-of-squares (SOS) constraints that can be written as a semidefinite program (SDP). This was done for each periodic domain of length L. The specific method used to produce the results in this section is described in Section B.1 in Appendix B.

Two families of carefully chosen modes were considered. These can be appreciated in Figure 4.1. One family is composed of six modes (boxed in blue) and the other is comprised of those six modes plus two more for a total of eight modes (boxed in red). Then the SDP was used to attempt to find a *quartic* Lyapunov function (as opposed to quadratic) for each of the two families.

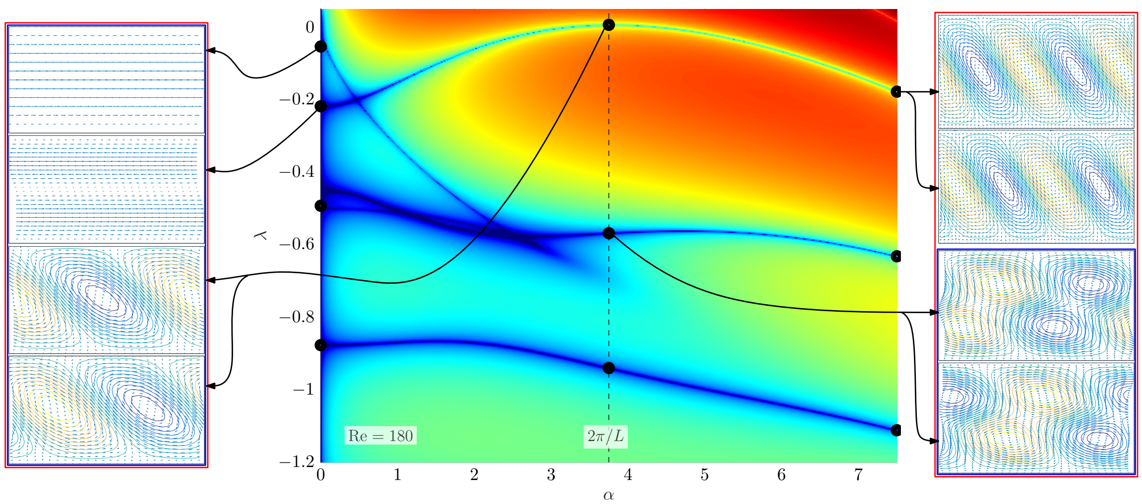

Figure 4.1: Determinant of the matrix associated to the eigenvalue problem. The branches in blue correspond to eigenvalues (where the determinant vanishes) with the exception of a triangle-looking shape in the upper left which corresponds to repeated roots (zero discriminant). The value of the Reynolds number is fixed (at 180), so the eigenvalues are a function of α. Each eigenvalue corresponding to $\alpha \neq 0$ has two eigenfunctions associated to it (shifted by a quarter period from each other). Two families of eigenmodes illustrated: six-mode family boxed in blue, and eight-mode family boxed in red.

The results can be observed in Figure 4.2, where the energy stability limit was shown for each periodic length L. The two curves above the energy stability limit are new larger lower bounds of the global stability limit, and for every Re under those curves the flow is globally asymptotically stable. For periodic lengths of under $L/h = 2.28$ the new lower bounds for Re_G are $\mathrm{Re}_{\mathrm{SOS},1} = 190$ using the six-mode family of eigenmodes and $\mathrm{Re}_{\mathrm{SOS},2} = 200$ using the eight-mode family of eigenmodes, which are both above the energy stability limit of $\mathrm{Re}_E = 177.2$ found by Orr over a century ago. This is the first improvement to the bound in that time-frame.

It should be noted that being able to find better bounds beyond the "bump" in the energy stability curve that occurs at about $L/h = 2.4$ is no easy task. The reason is that it would require using more modes (at least twelve), which make the computation much more expensive. At the current moment, the algorithms and solvers are definitely a limitation to getting more and better results. This is why posing the constraints in the more efficient manner possible is an important matter (see the Appendices).

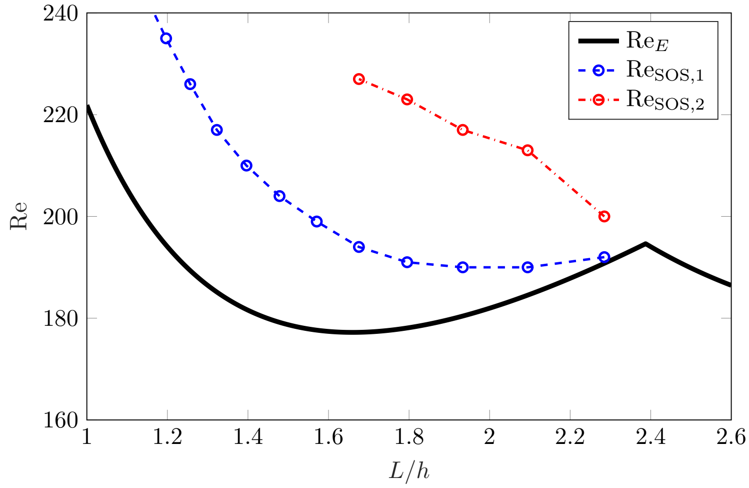

Figure 4.2: Energy stability limit (black) as a function of the periodic length of the domain. Larger lower bounds for the global stability limit resulting from the quartic Lyapunov functions found by the semidefinite program (SDP) with sum-of-squares (SOS) constraints for two families of energy eigenmodes: a six-mode family in blue and an eight-mode family in red.

5 Conclusions

Using new techniques from optimization, namely, the tractable imposition of sum-of-squares polynomial constraints in an optimization algorithm, it was possible to construct high-order quartic Lyapunov functions that proved the global stability of 2D plane Couette flow beyond the energy stability limit. This marks the first improvement in such a result in over a century! More importantly, the techniques can be utilized to analyze different flows. This will be left for future work. The current main limitation is computational power or better algorithms, and overcoming this limitation would allow to increase the number of energy eigenmodes to be included in the uncertain dynamical system that describes the fluid.

Acknowledgments

First, I wish to thank David Goluskin for supervising this project, for his support, and for sharing all his insight with me. I also wish to thank Giovanni Fantuzzi, who contributed some ideas in this work and was always open to bounce opinions back and forth. I also thank Sergei Chernyshenko for telling me about the program and encouraging me to apply. Lastly, I wish to thank all the GFD fellows, Claudia and Mary-Louise for organizing everything, and the whole environment at WHOI for making it a wonderful summer.

A Original Approach

First, I will give some general comments on the choice of $V(\mathbf{a}, q^2)$. Then, I will give the bounds derived for $\mathbf{\Theta_{ab}}(\mathbf{u_s}, \mathbf{a})$ and $\mathbf{\Theta_c}(\mathbf{u_s})$, and describe the usual choice of $\Xi(\mathbf{a}, q^2)$, which will be non-polynomial, and how it converts to a valid SDP.

A.1 Choice of Lyapunov function ansatz

The goal is for $V(\mathbf{a}, q^2)$ to be an SOS, where in this context a polynomial in $(\mathbf{a}, q^2)$ means a linear combination of monomials in the variables $\mathbf{a} = [a_1, \ldots, a_m]$ and even powers of q (i.e. monomials like a_1 of degree 1, $a_1 q^2$ of degree 3, $a_1 a_2 q^4$ of degree 6, etc.). Since $V(\mathbf{a}, q^2)$ will be constrained to be an SOS, it follows that the highest and lowest degree monomials in the linear combination must be of even degree. In particular, $\deg(V)$ must be even.

Additionally, we want $-(\tilde{G} + \Xi)$ to be an SOS too, so it must also have even degree. Here, Ξ is assumed to be positive definite (it is normally a positive bound of $\mathbf{M}\cdot(\mathbf{\Theta_{ab}}+\mathbf{\Theta_c})$). Thus, $\tilde{G}$ must be negative definite such that $\tilde{G} + \Xi$ is still negative definite. If $\deg(\Xi) > \deg(\tilde{G})$, then $\tilde{G} + \Xi$ will be positive for large $(\mathbf{a}, q^2)$ and this is precisely what we do *not* want. Therefore, assume $\deg(\tilde{G}) \geq \deg(\Xi)$, with $\deg(\tilde{G})$ being even (in order for it to be negative definite). The term $\frac{\partial V}{\partial \mathbf{a}} \cdot \mathbf{f}(\mathbf{a})$ in $\tilde{G}$ seems to be inconveniently of odd degree as $\mathbf{f}(\mathbf{a})$ is quadratic in $\mathbf{a}$ and $\deg(\frac{\partial V}{\partial \mathbf{a}})$ is odd (as $\deg(V)$ is even). To overcome this hurdle, a viable and elegant approach is to choose the component of highest degree in V to be a power of the kinetic energy $E = \frac{1}{2}(|\mathbf{a}|^2 + q^2)$, because with this choice, $\frac{\partial V}{\partial \mathbf{a}} \cdot \mathbf{f}(\mathbf{a})$ is of even degree (there are cancellations). Thus, the only component of degree $\deg(V)$ in V is chosen to be $(|\mathbf{a}|^2 + q^2)^{\deg(V)/2}$, and this ensures that $\tilde{G}$ and $\mathbf{M}$ are of even degree, with $\deg(V) = \deg(\tilde{G}) = \deg(\mathbf{M}) + 2$.

Next, we know $V(\mathbf{0}, 0) = 0$, so there must not be any constants in V. Since V will be chosen to be an SOS polynomial, this implies there must not be any linear terms in $a_1, \ldots, a_m$ either. As a result, in the interest of generality, V is assumed to be a linear combination of monomials in the variables $a_1, \ldots, a_m$ and q^2 of degree greater than or equal to 2 and less than or equal to $\deg(V) - 1$ along with the polynomial $(|\mathbf{a}|^2 + q^2)^{\deg(V)/2}$. That is, it should take the form,

$$V(\mathbf{a}, q^2) = \sum_{2 \leq \deg(\mathrm{mon}_\iota) \leq \deg(V)-1} c_\iota \mathrm{mon}_\iota(\mathbf{a}, q^2) + (|\mathbf{a}|^2 + q^2)^{\deg(V)/2}, \tag{A.1}$$

where ι indexes all monomials in the variables $a_1, \ldots, a_m$ and even powers of q, $\mathrm{mon}_\iota(\mathbf{a}, q^2)$, and the c_ι are unknown real coefficients associated to those monomials. Notice that the last homogeneous polynomial, $(|\mathbf{a}|^2 + q^2)^{\deg(V)/2}$, does not have a coefficient. This is because, in principle, V can be scaled by any constant, and this will not change the conclusion that $V > 0$ and that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$, so there is a freedom to fix at least one of the coefficients in V. Also, note that with this choice, $\tilde{G}$ does not have any constant or linear terms and $\mathbf{M}(\mathbf{0}, 0) = 0$.

Finally, placing the constraint that V should be an SOS polynomial is not sufficient, since this only ensures that V is positive semidefinite. To ensure that it is positive definite, a margin or barrier function must be added. That is, replace the condition $V(\mathbf{a}, q^2) > 0$, by $V(\mathbf{a}, q^2) \geq \varepsilon(|\mathbf{a}|^2 + q^2) > 0$. Thus, place the constraints

$$V(\mathbf{a}, q^2) - \varepsilon(|\mathbf{a}|^2 + q^2) \in \mathrm{SOS}(\mathbf{a}, q)\,, \qquad \frac{\partial V(\mathbf{a}, q^2)}{\partial q^2} \in \mathrm{SOS}(\mathbf{a}, q)\,, \tag{A.2}$$

where $\mathrm{SOS}(\mathbf{a}, q)$ is the set of sum-of-squares polynomials in the variables $a_1, \ldots, a_m$ and q.

A.2 Bounds of $\boldsymbol{\Theta}_{\mathbf{ab}}$ and $\boldsymbol{\Theta}_{\mathbf{c}}$

In [7, 9] the bounds of each component of $\boldsymbol{\Theta}_{\mathbf{ab}}$ and $\boldsymbol{\Theta}_{\mathbf{c}}$ were derived. Let $\tilde{\mathbf{a}} = [1, a_1, \ldots, a_m]$, indexed from 0 so that $\tilde{a}_0 = 1$ and $\tilde{a}_i = a_i$ for $i = 1, \ldots, m$. Then, $\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a}) = \langle \mathbf{u_s}\,, \mathbf{h}_{ij} \rangle \tilde{a}_j$, and the bounds are,

$$\begin{aligned} |\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})| &\leq \sqrt{\tilde{\mathbf{a}}^{\mathsf{T}} \mathbf{R}_i \tilde{\mathbf{a}} q^2}\,, &\qquad (R_i)_{kl} &= \langle \widetilde{\mathbf{h}}_{ik}\,, \widetilde{\mathbf{h}}_{il} \rangle\,, \\ |\Theta_{\mathbf{c}i}(\mathbf{u_s})| &\leq C_i q^2\,, & C_i &= \|\rho(\mathbf{D}_i)\|_\infty = \sup_{\mathbf{x}\in\Omega} \rho\big(\mathbf{D}_i(\mathbf{x})\big)\,, \end{aligned} \tag{A.3}$$

where $\mathbf{D}_i = \frac{1}{2}(\nabla \mathbf{u}_i + \nabla^{\mathsf{T}} \mathbf{u}_i)$, and $\rho(\mathbf{D}_i(\mathbf{x}))$ is the spectral radius of $\mathbf{D}_i(\mathbf{x})$. Here, $\widetilde{\mathbf{h}}_{ij}$ is the solenoidal projection (so $\nabla \cdot \widetilde{\mathbf{h}}_{ij} = 0$) of $\mathbf{h}_{ij}$ such that $\langle \widetilde{\mathbf{h}}_{ij}\,, \mathbf{u}_k \rangle = 0$ for all $k = 1, \ldots, m$ and satisfying that $\widetilde{\mathbf{h}}_{ij} \cdot \mathbf{n} = 0$ ($\mathbf{n}$ is the outer normal) wherever the perturbation velocity has no-slip boundary conditions. These bounds are important as they eliminate $\mathbf{u_s}$ and yield expressions only in terms of $a_1, \ldots, a_m$ and q^2.

A.3 A conservative bound of $\mathbf{M} \cdot (\boldsymbol{\Theta}_{\mathbf{ab}} + \boldsymbol{\Theta}_{\mathbf{c}})$

The original approach relies on first bounding with the absolute value, then using the triangle inequality twice, and lastly using the Cauchy-Schwarz inequality in $\mathbb{R}^m$,

$$\begin{aligned} \mathbf{M}(\mathbf{a}, q^2) \cdot \big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a}) + \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big) &\leq |\mathbf{M}(\mathbf{a}, q^2) \cdot \boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a})| + |\mathbf{M}(\mathbf{a}, q^2) \cdot \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})| \\ &\leq \sum_{i=1}^{m} |M_i(\mathbf{a}, q^2)| \big(|\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})| + |\Theta_{\mathbf{c}i}(\mathbf{u_s}, \mathbf{a})|\big) \\ &\leq |\mathbf{M}(\mathbf{a}, q^2)| \sqrt{\textstyle\sum_{i=1}^{m} \big(|\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})| + |\Theta_{\mathbf{c}i}(\mathbf{u_s}, \mathbf{a})|\big)^2}\,. \end{aligned} \tag{A.4}$$

Next, let $\delta_i \in (0, \infty)$ for all $i = 1, \ldots, m$, and note that

$$\begin{aligned} \sum_{i=1}^{m} (|\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})| + |\Theta_{\mathbf{c}i}(\mathbf{u_s})|)^2 &= \sum_{i=1}^{m} \Theta_{\mathbf{ab}i}^2(\mathbf{u_s}, \mathbf{a}) + 2|\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})|\,|\Theta_{\mathbf{c}i}(\mathbf{u_s})| + \Theta_{\mathbf{c}i}^2(\mathbf{u_s}) \\ &\leq \sum_{i=1}^{m} \Big((1 + \delta_i)\Theta_{\mathbf{ab}i}^2(\mathbf{u_s}, \mathbf{a}) + (1 + \tfrac{1}{\delta_i})\Theta_{\mathbf{c}i}^2(\mathbf{u_s})\Big) \\ &\leq \sum_{i=1}^{m} \big((1 + \delta_i)\tilde{\mathbf{a}}^{\mathsf{T}} \mathbf{R}_i \tilde{\mathbf{a}} q^2 + (1 + \tfrac{1}{\delta_i}) C_i^2 q^4\big) = p_{\boldsymbol{\Theta}}(\mathbf{a}, q^2)\,, \end{aligned} \tag{A.5}$$

where the so-called "Peter-Paul" inequality, $2w_1 w_2 \leq \delta w_1^2 + \frac{1}{\delta} w_2^2$ for any $\delta \in (0, \infty)$, is used. Hence,

$$\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a}) + \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big) \leq |\mathbf{M}(\mathbf{a}, q^2)| \sqrt{p_{\boldsymbol{\Theta}}(\mathbf{a}, q^2)} = \Xi(\mathbf{a}, q^2)\,. \tag{A.6}$$

Now we need to have that $\tilde{G} + \Xi < 0$ for all $(\mathbf{a}, q^2) \neq (\mathbf{0}, 0)$ (it is clear already that $\Xi(\mathbf{0}, 0) = 0$ because $p_{\mathbf{\Theta}}(\mathbf{0}, 0) = 0$). Unfortunately, Ξ is not polynomial (but $\Xi^2 = p_{\mathbf{\Theta}}|\mathbf{M}|^2 = p_{\mathbf{\Theta}}\mathbf{M}^\mathsf{T}\mathbf{M}$ is), so some manipulations are necessary to obtain a polynomial expression that is linear in the unknown coefficients c_ι of V (note that the c_ι are linearly present in $\tilde{G}$ and $\mathbf{M}$). Using Schur complements, the inequality can be rewritten as,

$$\tilde{G} + \Xi < 0 \quad \Leftrightarrow \quad 0 \leq \Xi < -\tilde{G} \quad \Leftrightarrow \quad \begin{cases} -\tilde{G} > 0 \\ p_{\mathbf{\Theta}}\tilde{G}^2 - p_{\mathbf{\Theta}}\Xi^2 > 0 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} -p_{\mathbf{\Theta}}\tilde{G} & p_{\mathbf{\Theta}}\mathbf{M}^\mathsf{T} \\ p_{\mathbf{\Theta}}\mathbf{M} & -\tilde{G}\mathbf{I} \end{bmatrix} \succ 0 \,. \tag{A.7}$$

Again, use a barrier function to obtain strict positivity, so that you should enforce the constraint

$$\begin{bmatrix} -p_{\mathbf{\Theta}}\tilde{G} & p_{\mathbf{\Theta}}\mathbf{M}^\mathsf{T} \\ p_{\mathbf{\Theta}}\mathbf{M} & -\tilde{G}\mathbf{I} \end{bmatrix} - \varepsilon(|\mathbf{a}|^2 + q^2) \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I} \end{bmatrix} \in \mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q) \,, \tag{A.8}$$

where $\mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q)$ is the set of sum-of-squares polynomial positive semidefinite matrices.

Another slight deviation of this condition is to use the barrier function beforehand, so that $\tilde{G} + \Xi + \varepsilon(|\mathbf{a}|^2 + q^2) \leq 0$, which results in the constraint

$$\begin{bmatrix} -p_{\mathbf{\Theta}}\big(\tilde{G} + \varepsilon(|\mathbf{a}|^2 + q^2)\big) & p_{\mathbf{\Theta}}\mathbf{M}^\mathsf{T} \\ p_{\mathbf{\Theta}}\mathbf{M} & -\big(\tilde{G} + \varepsilon(|\mathbf{a}|^2 + q^2)\big)\mathbf{I} \end{bmatrix} \in \mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q) \,. \tag{A.9}$$

Both (A.8) and (A.9) are "matrix" constraints of the form $\mathbf{T} \succeq 0$. These can be quite expensive to enforce. As an example, with only $m = 6$ modes and the fastest solver available, the SDP enforcing (A.2) and (A.8) took roughly 2 hours. However, $\mathbf{T}$ is a very sparse matrix so if you are careful, it is natural to expect some savings. The most natural approach is to add variables $\tilde{\mathbf{z}} = (z_0, z_1, \ldots, z_m)$, where $\mathbf{z} = (z_1, \ldots, z_m)$, and consider the equivalent statement $\tilde{\mathbf{z}}^\mathsf{T}\mathbf{T}\tilde{\mathbf{z}} \geq 0$. Due to the sparsity, (A.8) is rewritten as

$$-\big(p_{\mathbf{\Theta}}\tilde{G} + \varepsilon(|\mathbf{a}|^2 + q^2)\big)z_0^2 + 2z_0 p_{\mathbf{\Theta}}\mathbf{M}\cdot\mathbf{z} - \big(\tilde{G} + \varepsilon(|\mathbf{a}|^2 + q^2)\big)\mathbf{z}\cdot\mathbf{z} \in \mathrm{SOS}(\mathbf{a}, q, \tilde{\mathbf{z}}) \,. \tag{A.10}$$

With this formulation, the computation time was reduced to roughly 1 minute in the same machine. Similarly, (A.8) is rewritten as

$$-\,p_{\mathbf{\Theta}}\big(\tilde{G} + \varepsilon(|\mathbf{a}|^2 + q^2)\big)z_0^2 + 2z_0 p_{\mathbf{\Theta}}\mathbf{M}\cdot\mathbf{z} - \big(\tilde{G} + \varepsilon(|\mathbf{a}|^2 + q^2)\big)\mathbf{z}\cdot\mathbf{z} \in \mathrm{SOS}(\mathbf{a}, q, \tilde{\mathbf{z}}) \,. \tag{A.11}$$

Lastly, an alternative to exploit the sparsity of $\mathbf{T}$ (and does not require extra variables) is to use Agler's theorem as proposed in [5], but we have not implemented it in this note.

The SDP enforcing the constraints (A.2) and (A.10) is called "Original 1", while the SDP enforcing the constraints (A.2) and (A.11) is called "Original 2",

$$\begin{aligned} &\max \varepsilon \,, \quad \text{subject to} \quad (\text{A.2}) \;\&\; (\text{A.10}) \quad \leftarrow \quad \text{Original 1} \,, \\ &\max \varepsilon \,, \quad \text{subject to} \quad (\text{A.2}) \;\&\; (\text{A.11}) \quad \leftarrow \quad \text{Original 2} \,. \end{aligned} \tag{A.12}$$

In both cases ε is maximized. If $\max \varepsilon$ is ultimately positive, then the problem is feasible, and otherwise it is infeasible. As expected, both methods above seem to behave very similarly and have the same limitations, but Original 2 produces larger values of ε which are safer to trust. Unfortunately, adding just two more modes ($m = 8$) has huge memory requirements for solvers using interior point methods, so no simulations could be completed. More detailed results are found in Section E.

B A New Family of Methods

This family of methods relies on a better bound for $\mathbf{M} \cdot (\mathbf{\Theta_{ab}} + \mathbf{\Theta_c})$, which still uses (A.4), but considers the bound before the Cauchy-Schwarz inequality is applied,

$$\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\mathbf{\Theta_{ab}}(\mathbf{u_s}, \mathbf{a}) + \mathbf{\Theta_c}(\mathbf{u_s})\big) \le \sum_{i=1}^{m} |M_i(\mathbf{a}, q^2)| \big(|\Theta_{\mathbf{ab}i}(\mathbf{u_s}, \mathbf{a})| + |\Theta_{\mathbf{c}i}(\mathbf{u_s})|\big) . \tag{B.1}$$

The results of the methods about to be described can be found in Section E.

B.1 Method 1

Consider first

$$|M_i|\,|\Theta_{\mathbf{ab}i}| \le |M_i|\sqrt{\tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2} \le r_i \,, \tag{B.2}$$

where (A.3) was used and where $r_i(\mathbf{a}, q^2)$ is an unknown polynomial of the form,

$$r_i(\mathbf{a}, q^2) = \sum_{2 \le \deg(\mathrm{mon}_\iota) \le \deg(V)} c^r_\iota \mathrm{mon}_\iota(\mathbf{a}, q^2) \,, \tag{B.3}$$

where the coefficients, c^r_ι are unknown. Notice that the constant and linear terms are eliminated because we eventually want $\Xi(\mathbf{0}, 0) = 0$. Also, we want that $\deg(V) = \deg(\tilde{G}) \ge \deg(\Xi) \ge \deg(r_i)$, so it follows $\deg(r_i) \le \deg(V)$. Additionally, $\deg(r_i)$ should be an even power strictly larger than $\deg(M_i) = \deg(V) - 2$ (see (B.2)). The only choice is then to set $\deg(r_i) = \deg(V)$ and this explains the expression in (B.3). Proceeding as with (A.7) now yields,

$$|M_i|\sqrt{\tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2} \le r_i \quad \Leftrightarrow \quad \begin{bmatrix} \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 r_i & \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 M_i \\ \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 M_i & r_i \end{bmatrix} \succeq 0 \,, \tag{B.4}$$

for each $i = 1, \ldots, m$. Therefore, add the SOS constraints,

$$\begin{bmatrix} \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 r_i & \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 M_i \\ \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 M_i & r_i \end{bmatrix} \in \mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q) \,, \qquad \forall i = 1, \ldots, m \,. \tag{B.5}$$

Next, using (A.3) again, look at

$$|M_i|\,|\Theta_{\mathbf{c}i}| \le |M_i| C_i q^2 \le s_i C_i q^2 \,, \tag{B.6}$$

where $s_i(\mathbf{a}, q^2)$ is an unknown polynomial of the form,

$$s_i(\mathbf{a}, q^2) = \sum_{0 \le \deg(\mathrm{mon}_\iota) \le \deg(V)-2} c^s_\iota \mathrm{mon}_\iota(\mathbf{a}, q^2) \,, \tag{B.7}$$

where the coefficients, c^s_ι are unknown. Notice that $\deg(s_i) = \deg(V) - 2 = \deg(\mathbf{M})$, because we want that $\deg(V) = \deg(\tilde{G}) \ge \deg(\Xi) \ge \deg(s_i C_i q^2)$ (so $\deg(s_i)$ cannot be any larger than $\deg(V) - 2$). Now, the condition can be rewritten as

$$|M_i| \le s_i \quad \Leftrightarrow \quad \begin{cases} M_i \le s_i \\ -s_i \le M_i \end{cases} , \tag{B.8}$$

for each $i = 1, \ldots, m$. Therefore, add the SOS constraints,

$$\begin{cases} s_i - M_i \in \mathrm{SOS}(\mathbf{a}, q) \\ s_i + M_i \in \mathrm{SOS}(\mathbf{a}, q) \end{cases}, \qquad \forall i = 1, \ldots, m\,. \tag{B.9}$$

Lastly, add (B.2) and (B.6) across all $i = 1, \ldots, m$ to yield,

$$\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a}) + \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big) \leq \sum_{i=1}^{m} \big(r_i(\mathbf{a}, q^2) + s_i(\mathbf{a}, q^2) C_i q^2\big) = \Xi(\mathbf{a}, q^2)\,. \tag{B.10}$$

By design, $\Xi(\mathbf{0}, 0) = 0$ and in general $\deg(\Xi) = \deg(\tilde{G}) = \deg(V)$. It only remains to enforce the condition that $\tilde{G} + \Xi < 0$, which in this case is very simple by imposing the SOS constraint,

$$-\tilde{G} - \Xi - \varepsilon(|\mathbf{a}|^2 + q^2) \in \mathrm{SOS}(\mathbf{a}, q)\,. \tag{B.11}$$

The final SDP takes the form,

$$\max \varepsilon\,, \qquad \text{subject to} \quad (\text{A.2}),\ (\text{B.5}),\ (\text{B.9}) \ \ \& \ \ (\text{B.11}) \qquad \leftarrow \quad \text{Method 1}\,. \tag{B.12}$$

B.2 Method 2

This time, let $\delta_i \in (0, \infty)$ for all $i = 1, \ldots, m$, and proceed as in (A.5) (but with each component separately and taking square roots a posteriori), so that

$$|M_i|\big(|\Theta_{\mathbf{ab}i}| + |\Theta_{\mathbf{c}i}|\big) \leq |M_i|\big(\sqrt{\tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2} + C_i q^2\big) \leq |M_i| \underbrace{\sqrt{(1+\delta_i)\tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2 + (1 + \tfrac{1}{\delta_i})C_i^2 q^4}}_{\sqrt{d_i(\mathbf{a}, q^2)}} \leq r_i\,, \tag{B.13}$$

where $r_i(\mathbf{a}, q^2)$ is an unknown polynomial with an ansatz as in (B.3). Manipulating as in (B.4), yields the SOS constraints

$$\begin{bmatrix} d_i r_i & d_i M_i \\ d_i M_i & r_i \end{bmatrix} \in \mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q)\,, \qquad \forall i = 1, \ldots, m\,. \tag{B.14}$$

Adding (B.13) among all $i = 1, \ldots, m$ gives,

$$\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a}) + \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big) \leq \sum_{i=1}^{m} r_i(\mathbf{a}, q^2) = \Xi(\mathbf{a}, q^2)\,, \tag{B.15}$$

and the remaining SOS constraint is

$$-\tilde{G} - \Xi - \varepsilon(|\mathbf{a}|^2 + q^2) \in \mathrm{SOS}(\mathbf{a}, q)\,, \tag{B.16}$$

with the final SDP taking the form,

$$\max \varepsilon\,, \qquad \text{subject to} \quad (\text{A.2}),\ (\text{B.14}) \ \ \& \ \ (\text{B.16}) \qquad \leftarrow \quad \text{Method 2}\,. \tag{B.17}$$

B.3 Method 3

This method aims to precompute bounds of each $|\Theta_{\mathbf{ab}i}|$ and then use the bounds of $|M_i|$ via the constraints in (B.9). It is supposed to produce a significant speed up in the computations by avoiding constraints of the form of (B.5) in the global (large) problem, and instead tackling these type of constraints in a previous step consisting of a series of much cheaper and easier small problems that precompute particular bounds. The crux is to develop a viable method to precompute these bounds and to find good criteria to have the most effective bounds possible.

First, consider bounds

$$0 \leq |\Theta_{\mathbf{ab}i}| \leq \sqrt{\tilde{\mathbf{a}}^{\mathsf{T}}\mathbf{R}_i\tilde{\mathbf{a}}q^2} \leq b_i\,, \tag{B.18}$$

with $b_i(\mathbf{a}, q^2)$ being an unknown homogeneous quadratic polynomial of the form,

$$b_i(\mathbf{a}, q^2) = \sum_{\deg(\mathrm{mon}_\iota)=2} c_\iota^b \mathrm{mon}_\iota(\mathbf{a}, q^2)\,, \tag{B.19}$$

where the coefficients, c_ι^b, are unknown. Note the ansatz forces $b_i(\mathbf{0}, 0) = 0$ (so that later on $\Xi(\mathbf{0}, 0) = 0$ as well). This is equivalent to the SOS constraints

$$\begin{bmatrix} \tilde{\mathbf{a}}^{\mathsf{T}}\mathbf{R}_i\tilde{\mathbf{a}}q^2 b_i & \tilde{\mathbf{a}}^{\mathsf{T}}\mathbf{R}_i\widetilde{\mathbf{a}}q^2 \\ \tilde{\mathbf{a}}^{\mathsf{T}}\mathbf{R}_i\tilde{\mathbf{a}}q^2 & b_i \end{bmatrix} \in \mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q)\,, \qquad \forall i = 1, \ldots, m\,. \tag{B.20}$$

The idea is to precompute the b_i beforehand under some optimization criterion (we have to choose what to minimize and maximize). Then, the c_ι^b coefficients will be known, and it is valid to use

$$\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a}) + \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big) \leq \sum_{i=1}^{m} s_i(\mathbf{a}, q^2)\big(b_i(\mathbf{a}, q^2) + C_i q^2\big) = \Xi(\mathbf{a}, q^2)\,, \tag{B.21}$$

where the s_i bound $|M_i|$ as in (B.6). This may look viable if one proceeds as in the previous two methods, but it is actually impossible.

To see the problem, simply focus on the assumed bound (B.18), and notice it is impossible to find such a bound. Indeed, assuming $\mathbf{a} = \mathbf{0}$ and $q \neq 0$ (recall $\widetilde{\mathbf{a}}$ has a constant nonzero component) it follows

$$\sqrt{\tilde{\mathbf{a}}^{\mathsf{T}}\mathbf{R}_i\tilde{\mathbf{a}}q^2} = \sqrt{D_i q^2} = \sqrt{D_i}\,|q|\,, \qquad \text{if } \mathbf{a} = \mathbf{0}\,, \tag{B.22}$$

where $D_i > 0$ is a constant. It is impossible to bound this positive function with a positive quadratic polynomial that passes through $q = 0$ (see the behaviour near $q = 0$).

To fix this issue, note that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ if and only if $\frac{\mathrm{d}V}{\mathrm{d}t}(|\mathbf{a}|^2 + q^2)^{k_E} < 0$ for any positive integer $k_E \in \mathbb{N}$. Thus, let $2E(\mathbf{a}, q^2) = |\mathbf{a}|^2 + q^2$, choose some $k_E \in \mathbb{N}$, and as in (2.9), note that

$$\frac{\mathrm{d}V}{\mathrm{d}t}\big(2E(\mathbf{a}, q^2)\big)^{k_E} \leq \tilde{G}(\mathbf{a}, q^2)\big(2E(\mathbf{a}, q^2)\big)^{k_E} + \tilde{\Xi}(\mathbf{a}, q^2)\,, \tag{B.23}$$

as long as $\frac{\partial V}{\partial q^2} \geq 0$ and that there exists $\tilde{\Xi}(\mathbf{a}, q^2)$ such that $\tilde{\Xi}(\mathbf{0}, 0) = 0$ and

$$\mathbf{M}(\mathbf{a}, q^2) \cdot \big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s}, \mathbf{a}) + \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big) \cdot \big(2E(\mathbf{a}, q^2)\big)^{k_E} \leq \tilde{\Xi}(\mathbf{a}, q^2)\,. \tag{B.24}$$

Thus, if such a $\tilde{\Xi}$ exists, $\frac{\partial V}{\partial q^2} \geq 0$, and $\tilde{G}(2E)^{k_E} + \tilde{\Xi} < 0$ it follows that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$.

To find such a $\tilde{\Xi}$, instead of (B.18), consider the bounds

$$|\Theta_{\mathbf{ab}i}|(2E)^{k_E} \leq \sqrt{\tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2}(2E)^{k_E} \leq b_i\,, \tag{B.25}$$

with $b_i(\mathbf{a}, q^2)$ being an unknown polynomial of the form,

$$b_i(\mathbf{a}, q^2) = \sum_{2\leq \deg(\mathrm{mon}_\iota)\leq 2(k_E+1)} c_\iota^b \mathrm{mon}_\iota(\mathbf{a}, q^2)\,, \tag{B.26}$$

where the coefficients, c_ι^b, are unknown. These bounds are now truly viable to find, since the situation in (B.22) no longer holds (i.e., when $\mathbf{a} = \mathbf{0}$ and $k_E = 1$ the function now grows cubically about the origin and this can be bounded by a positive quartic polynomial passing through $q = 0$). As before, (B.25) is equivalent to the SOS constraints,

$$\begin{bmatrix} \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2(2E)^{2k_E}b_i & \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2(2E)^{2k_E} \\ \tilde{\mathbf{a}}^\mathsf{T}\mathbf{R}_i\tilde{\mathbf{a}}q^2(2E)^{2k_E} & b_i \end{bmatrix} \in \mathrm{SOS}_{\mathbb{M}}(\mathbf{a}, q)\,, \qquad \forall i = 1, \ldots, m\,. \tag{B.27}$$

Making m SOS feasibility tests could provide the precomputed bounds b_i,

$$\text{Check feasibility of} \quad \text{(B.27)} \quad \leftarrow \quad \text{Precompute bound } i\,, \qquad \forall i = 1, \ldots, m\,, \tag{B.28}$$

which yields the coefficients, c_ι^b for each $i = 1, \ldots, m$.

Then, $\tilde{\Xi}$ is simply,

$$\mathbf{M} \cdot \big(\mathbf{\Theta_{ab}} + \mathbf{\Theta_c}\big) \cdot (2E)^{k_E} \leq \sum_{i=1}^{m} s_i\big(b_i + C_i q^2(2E)^{k_E}\big) = \tilde{\Xi}\,, \tag{B.29}$$

and the SOS constraint that implies $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ is

$$-\,(\tilde{G}(2E)^{k_E} + \tilde{\Xi}) - \varepsilon(|\mathbf{a}|^2 + q^2) \in \mathrm{SOS}(\mathbf{a}, q)\,. \tag{B.30}$$

Lastly, this method would consist of the SDP

$$\max \varepsilon\,, \qquad \text{subject to} \quad \text{(A.2)}, \ \text{(B.9)} \ \ \& \ \ \text{(B.30)} \quad \leftarrow \quad \text{Method 3}\,, \tag{B.31}$$

provided the m bounds in (B.25) have been precomputed using (B.28) in a previous step.

The feasibility test in (B.28) is by no means optimal in the sense that one would want the smallest possible upper bound in (B.25), but (B.28) only provides one such upper bound, which could be huge. Ideally, it could be useful to modify the ansatz for b_i in (B.26) to include an optimization parameter that somehow ensures that b_i is as small as possible. In essence, one should try to change (B.28) from a feasibility problem to an intelligently chosen optimization problem. There are many ways to do this, but no more details are given here for the time being, since the method has not been implemented yet.

From experience, we expect this method to be two or three orders of magnitude faster than the previous two methods. Once we implement this method, more details will be given.

C Improving the Bounds Even More

Now look at (A.4) once again, but this time stop right after using the triangle inequality for the first time,

$$\mathbf{M}(\mathbf{a},q^2)\cdot\big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s},\mathbf{a})+\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big)\leq|\mathbf{M}(\mathbf{a},q^2)\cdot\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s},\mathbf{a})|+|\mathbf{M}(\mathbf{a},q^2)\cdot\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})|\,. \tag{C.1}$$

C.1 Bound of $|\mathbf{M}\cdot\boldsymbol{\Theta}_{\mathbf{ab}}|$

Proceed as in [2] and using the summation convention note that

$$|\mathbf{M}\cdot\boldsymbol{\Theta}_{\mathbf{ab}}|=|\langle\mathbf{u_s}\,,M_i\widetilde{\mathbf{h}}_{ij}\tilde{a}_j\rangle|\leq\|M_i\widetilde{\mathbf{h}}_{ij}\tilde{a}_j\|\,|q|\,. \tag{C.2}$$

In [2], (31) is incorrect (it is even dependent on $\mathbf{x}\in\Omega$), because this an L^2 norm and must be computed explicitly. It can be written as,

$$\|M_i\widetilde{\mathbf{h}}_{ij}\tilde{a}_j\|^2=\underbrace{\begin{bmatrix}\widetilde{\mathbf{a}}_1^{\square}\\ \vdots\\ \widetilde{\mathbf{a}}_m^{\square}\end{bmatrix}^{\mathsf{T}}}_{(\mathbf{a}^{\mathbf{M}})^{\mathsf{T}}}\underbrace{\begin{bmatrix}\mathbf{H}_{11}^{\square}&\cdots&\mathbf{H}_{1m}^{\square}\\ \vdots&\ddots&\vdots\\ \mathbf{H}_{m1}^{\square}&\cdots&\mathbf{H}_{mm}^{\square}\end{bmatrix}}_{\mathbf{H}}\underbrace{\begin{bmatrix}\widetilde{\mathbf{a}}_1^{\square}\\ \vdots\\ \widetilde{\mathbf{a}}_m^{\square}\end{bmatrix}}_{\mathbf{a}^{\mathbf{M}}}\,,\quad\widetilde{\mathbf{a}}_i^{\square}=M_i\widetilde{\mathbf{a}}=M_i\begin{bmatrix}1\\ a_1\\ \vdots\\ a_m\end{bmatrix}\,,\quad(H_{ij}^{\square})_{kl}=\langle\widetilde{\mathbf{h}}_{ik}\,,\widetilde{\mathbf{h}}_{jl}\rangle\,, \tag{C.3}$$

so that $\mathbf{H}\in\mathbb{R}^{m(m+1)\times m(m+1)}$ is a positive semidefinite matrix (compare to the much smaller matrices $\mathbf{R}_i\in\mathbb{R}^{(m+1)\times(m+1)}$), and where it is clear that $\mathbf{H}_{ii}^{\square}=\mathbf{R}_i$ for every $i=1,\dots,m$. Thus,

$$|\mathbf{M}\cdot\Theta_{\mathbf{ab}i}|\leq\sqrt{(\mathbf{a}^{\mathbf{M}})^{\mathsf{T}}\mathbf{H}\mathbf{a}^{\mathbf{M}}q^2}=\sqrt{(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)^{\mathsf{T}}(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)}\,, \tag{C.4}$$

where $\mathbf{H}^{1/2}=(\mathbf{H}^{1/2})^{\mathsf{T}}$ is the unique positive semidefinite square root of $\mathbf{H}$ (computed using the eigenvalue decomposition). Note that a unique Cholesky decomposition also exists and could be used, but the typical algorithm breaks down in the semidefinite case, so it is preferable to use the usual square root of the matrix.

To give a rough idea of how to deal with this bound, assume $\mathbf{M}\cdot\boldsymbol{\Theta}_{\mathbf{c}}$ has been bounded as in (B.6)–(B.9) from Method 1, so that

$$\mathbf{M}(\mathbf{a},q^2)\cdot\big(\boldsymbol{\Theta}_{\mathbf{ab}}(\mathbf{u_s},\mathbf{a})+\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{u_s})\big)\leq\sqrt{(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)^{\mathsf{T}}(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)}+\sum_{i=1}^{m}s_i(\mathbf{a},q^2)C_iq^2=\Xi(\mathbf{a},q^2)\,. \tag{C.5}$$

Then, to enforce $\tilde{G}+\Xi<0$, note that

$$\begin{aligned}\tilde{G}+\Xi<0\quad&\Leftarrow\quad\underbrace{\tilde{G}+\sum_{i=1}^{m}s_i(\mathbf{a},q^2)C_iq^2+\varepsilon(|\mathbf{a}|^2+q^2)}_{\tilde{G}_0}+\sqrt{(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)^{\mathsf{T}}(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)}\leq 0\\ &\Leftrightarrow\begin{bmatrix}-\tilde{G}_0&(\mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q)^{\mathsf{T}}\\ \mathbf{H}^{1/2}\mathbf{a}^{\mathbf{M}}q&-\tilde{G}_0\mathbf{I}\end{bmatrix}\succeq 0\,.\end{aligned} \tag{C.6}$$

This formulation would have been impossible without the decomposition in (C.4) which uses $\mathbf{H}^{1/2}$, since $(\mathbf{a}^{\mathbf{M}})^{\intercal}\mathbf{H}\mathbf{a}^{\mathbf{M}}q^2$ is quadratic in $\mathbf{M}$ and thus quadratic in the unknown coefficients c_ι of V. Thankfully, this reformulation avoids any such problems. However, one should note that the matrix in (C.6) is huge. Therefore, creating auxiliary variables $z_0, z_1, \dots, z_{m(m+1)}$ to exploit the sparsity might be computationally prohibitive. Perhaps using Agler's theorem as described in [5] is a more viable approach. In any case, this formulation was discovered at the last moment, so it has not been implemented.

C.2 Bound of $|\mathbf{M} \cdot \mathbf{\Theta_c}|$

The best one could hope for is to proceed as in [9] but with the whole $\mathbf{M} \cdot \mathbf{\Theta_c}$ instead. This yields,

$$|\mathbf{M} \cdot \mathbf{\Theta_c}| \leq \left\| \rho\Big(\textstyle\sum_{i=1}^m M_i \mathbf{D}_i\Big) \right\|_\infty = \sup_{\mathbf{x}\in\Omega} \rho\Big(\textstyle\sum_{i=1}^m M_i \mathbf{D}_i\Big) . \tag{C.7}$$

Unfortunately we have not found a way to compute this quantity, even if M_i was known, which in principle it is not. The best bounds at this moment are those calculated via (B.6)–(B.9) in Method 1.

D Global Stability Using Bounds and Conditional Stability

This approach was proposed in [2] very briefly, so here I will give some more details. As a heads up, it is more expensive computationally, but, in principle, still worth trying if it allows to prove global stability beyond what the other methods can. It is divided into three steps:

1. Choose a positive definite quantity of interest $\Phi(\mathbf{a}, q^2) \geq 0$, so that $\Phi(\mathbf{a}, q^2) = 0$ if and only if $(\mathbf{a}, q^2) = (\mathbf{0}, 0)$, and which you expect to be conditionally stable for the known equilibrium point of the dynamical system, $(\mathbf{0}, 0)$.
2. Compute a time-average bound $U \in \mathbb{R}$ such that $\overline{\Phi(\mathbf{a}, q^2)} \leq U$, where $\overline{\Phi(\mathbf{a}, q^2)} = \lim_{T\to\infty} \frac{1}{T}\int_0^t \Phi(\mathbf{a}(t), q^2(t))\,\mathrm{d}t$. Obviously this implies that for each initial condition $(\mathbf{a}(0), q^2(0))$ there exists some instant t^*, such that $\Phi(\mathbf{a}(t^*), q^2(t^*)) = U$.
3. Show that whenever $0 < \Phi < C = U + \bar{\varepsilon}$, then $\frac{\mathrm{d}\Phi}{\mathrm{d}t} < 0$. This implies that if $\Phi(\mathbf{a}, q^2) < C$ at some instant t, then $\lim_{t\to\infty} \Phi(\mathbf{a}(t), q^2(t)) = 0$ and as a result $\lim_{t\to\infty}(\mathbf{a}(t), q^2(t)) = (\mathbf{0}, 0)$. Here, $\bar{\varepsilon} > 0$ is a fixed positive small quantity. In particular, by step 2, it follows that regardless of the initial condition $\lim_{t\to\infty}(\mathbf{a}(t), q^2(t)) = (\mathbf{0}, 0)$, which means that the flow is globally stable (since the velocity satifies $\|\mathbf{u}(t)\|^2 = |\mathbf{a}(t)|^2 + q^2(t)$).

An important comment on step 3, is that it is *not* sufficient to simply prove that $\frac{\mathrm{d}\Phi}{\mathrm{d}t} < 0$ in an open set (like any ball of radius R) containing $(\mathbf{a}, q^2) = (\mathbf{0}, 0)$, as this will only imply that $(\mathbf{0}, 0)$ is locally asymptotically stable, but will not give any information on the critical set (which will be some unknown ball of radius $0 < R_0 < R$ if your initial set was a ball of radius R) leading to asymptotic convergence. You will only know that such a critical set around $(\mathbf{0}, 0)$ exists and is a subset of the original open set. So it is fundamental to prove that $\frac{\mathrm{d}\Phi}{\mathrm{d}t} < 0$ whenever $\Phi < C$, since this actually does imply that the critical set are the points for which $\Phi < C$.

The combination of the three steps will be referred to as the "Bounding" method for short.

D.1 Step 1

The most natural choice for a positive-definite Φ, which you expect to be conditionally stable, is precisely the Lyapunov function associated to the linearized system of equations. For a truncated set of m modes, $\mathbf{a} = (a_1, \ldots, a_m)$, and forgetting about $\mathbf{u_s}$, the linearized dynamical system (see (2.5) and (2.6)) is simply,

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \mathbf{f}(\mathbf{a}) = \mathbf{L}\mathbf{a}\,, \qquad f_i(\mathbf{a}) = \big(\tfrac{1}{\mathrm{Re}}\langle \mathbf{u}_i\,, \nabla^2\mathbf{u}_j\rangle - \langle \mathbf{u}_i\,, \mathbf{u}_j\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{u}_j\rangle\big)a_j = L_{ij}a_j\,. \tag{D.1}$$

To find a Lyapunov function simply seek a positive definite matrix $\mathbf{P} \succ 0$ such that $\mathbf{L}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\mathbf{L} \prec 0$, and the Lyapunov function will be $\Phi_T(\mathbf{a}) = \mathbf{a}^{\mathsf{T}}\mathbf{P}\mathbf{a}$. As usual to ensure strict positivity or negativity one needs barrier functions, so that one can solve an SDP enforcing,

$$\max \varepsilon\,, \qquad \text{subject to} \quad \mathbf{P} - \varepsilon\mathbf{I} \succeq 0 \qquad -\big(\mathbf{L}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\mathbf{L}\big) - \varepsilon\mathbf{I} \succeq 0, \qquad \mathrm{tr}(\mathbf{P}) = m\,, \tag{D.2}$$

where $\mathrm{tr}(\mathbf{P})$ is the usual trace of a matrix. The last condition simply ensures the scaling of $\mathbf{P}$ is fixed, and also has the nice property that $|\mathbf{a}|^2$ also satisfies it. The maximization of ε intuitively ensures that $\Phi_T(\mathbf{a}) = \mathbf{a}^{\mathsf{T}}\mathbf{P}\mathbf{a}$ is as far away from zero as possible, and thus more likely to have better conditional stability behavior.

However, the role of q^2 must also be added. We decided to do this a posteriori by simply considering,

$$\Phi(\mathbf{a}, q^2) = \mathbf{a}^{\mathsf{T}}\mathbf{P}\mathbf{a} + \alpha_\Phi q^2\,, \tag{D.3}$$

where $\alpha_\Phi > 0$ is a constant to be chosen. Due to the constraint that $\mathrm{tr}(\mathbf{P}) = m$, it makes sense (in terms of order of magnitude) to choose $\alpha_\Phi = 1$, but other possibilities include $\alpha_\Phi = \min_i P_{ii}$ or $\alpha_\Phi = \max_i P_{ii}$. In this work we chose $\alpha_\Phi = 1$.

D.2 Step 2

Here, we proceed as described in [6], where it was shown that for bounded trajectories (in a fluid system all trajectories are bounded) and for any *storage* function $V(\mathbf{a}, q^2)$,

$$\frac{\mathrm{d}V(\mathbf{a}(t), q^2(t))}{\mathrm{d}t} + \Phi(\mathbf{a}, q^2) \le U \quad \Rightarrow \quad \overline{\Phi(\mathbf{a}, q^2)} \le U\,. \tag{D.4}$$

Looking at (2.9), it is clear that it suffices to show that

$$\tilde{G} + \Xi + \Phi \le U\,. \tag{D.5}$$

Here, once again Ξ is simply a bound in terms of $(\mathbf{a}, q^2)$, and there are several ways to obtain a valid Ξ, as we have shown throughout this document. We chose the one from Method 1, as it is the one of the tightest bounds leading to a problem of reasonable size and which does not use the "Peter-Paul" inequality. Thus, choose Ξ as in (B.10), so that the SDP becomes

$$\min U\,, \qquad \text{subject to} \quad \frac{\partial V}{\partial q^2} \in \mathrm{SOS}(\mathbf{a}, q),\ \text{(B.5)},\ \text{(B.9)}\ \ \&\ \ U - \tilde{G} - \Xi - \Phi \in \mathrm{SOS}(\mathbf{a}, q)\,. \tag{D.6}$$

The main difference in these constraints and derivation (with respect to the other methods) is the presence of U, that V no longer has to be positive definite (and is no longer free of scaling) and more importantly that $V(\mathbf{0},0)$ and $\Xi(\mathbf{0},0)$ no longer have to vanish. Therefore, the ansatzes (A.1) and (B.3) should be modified to

$$\begin{aligned} V(\mathbf{a},q^2) &= \sum_{0\le\deg(\text{mon}_\iota)\le\deg(V)-1} c_\iota \text{mon}_\iota(\mathbf{a},q^2) + c_E(|\mathbf{a}|^2+q^2)^{\deg(V)/2}\,, \\ r_i(\mathbf{a},q^2) &= \sum_{0\le\deg(\text{mon}_\iota)\le\deg(V)} c_\iota^r \text{mon}_\iota(\mathbf{a},q^2)\,. \end{aligned} \tag{D.7}$$

D.3 Step 3

We know that $\Phi(\mathbf{0},0)=0$ and that $\frac{\mathrm{d}\Phi}{\mathrm{d}t}(\mathbf{0},0)=0$, so when $(\mathbf{a},q^2)\neq(\mathbf{0},0)$, the final step is to prove that $\Phi<C=U+\bar\varepsilon$ implies that $\frac{\mathrm{d}\Phi}{\mathrm{d}t}<0$, where $\bar\varepsilon$ is a small positive number. Using the S-procedure, it is sufficient to satisfy the condition

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}\le-(C-\Phi)S\,, \tag{D.8}$$

where S is a positive definite function satisfying that $S(\mathbf{0},0)=0$ (so that $\frac{\mathrm{d}\Phi}{\mathrm{d}t}(\mathbf{0},0)=0$ does not violate the inequality). Note, all the focus here is the behavior near the origin $(\mathbf{0},0)$ since far away, where $-(C-\Phi)S$ is very positive, it does not really matter.

Next, simply treat Φ as V in (2.9) and proceed analogously to note that it is sufficient to prove that

$$\tilde G_\Phi+\Xi_\Phi\le-(C-\Phi)S\,, \tag{D.9}$$

where $\tilde G_\Phi=\frac{\partial\Phi}{\partial\mathbf{a}}\cdot\mathbf{f}(\mathbf{a})+2\frac{\partial\Phi}{\partial q^2}\kappa q^2$ and where $\mathbf{M}_\Phi\cdot(\mathbf{\Theta_{ab}}+\mathbf{\Theta_c})\le\Xi_\Phi$ with $\mathbf{M}_\Phi=\frac{\partial\Phi}{\partial\mathbf{a}}-2\frac{\partial\Phi}{\partial q^2}\mathbf{a}$ and $\Xi_\Phi(\mathbf{0},0)=0$. Note that the inequality is sufficient since we already know that $\frac{\partial\Phi}{\partial q^2}=\alpha_\Phi>0$ by construction (see (D.3)). Once again, Ξ_Φ can be estimated in different ways, and for much the same reasons described in step 2 we chose the technique from Method 1.

Since S is unknown, expected to be an SOS polynomial and with $S(\mathbf{0},0)=0$, consider the ansatz

$$S(\mathbf{a},q^2)=\sum_{2\le\deg(\text{mon}_\iota)\le\deg(S)} c_\iota^S\text{mon}_\iota(\mathbf{a},q^2)\,, \tag{D.10}$$

where $\deg(S)$ is an even number to be chosen freely. Therefore, the final SDP is a feasibility test of the following conditions,

$$S-\bar{\bar\varepsilon}(|\mathbf{a}|^2+q^2)\in\text{SOS}(\mathbf{a},q),\ (\text{B.5})_\Phi,\ (\text{B.9})_\Phi\ \ \&\ \ -\tilde G_\Phi-\Xi_\Phi-(C-\Phi)S\in\text{SOS}(\mathbf{a},q)\,, \tag{D.11}$$

where $\bar{\bar\varepsilon}$ is another small positive number. One could attempt to maximize $\bar{\bar\varepsilon}$, but in this particular case does not yield interesting insight and seems to add cost. Here, $(\text{B.5})_\Phi$ and $(\text{B.9})_\Phi$ are simply (B.5) and (B.9) but with $\mathbf{M}$ replaced by $\mathbf{M}_\Phi$ instead. Meanwhile in the ansatzes (B.3) and (B.7), $\deg(V)$ should be replaced by 4, since $\deg(\mathbf{M}_\Phi)$ is not $\deg(\Phi)-2=0$ ($\deg(\mathbf{M})=\deg(V)-2$ only happens due to the ansatz for V in (B.3)), but rather $\deg(\mathbf{M}_\Phi)=1$ so the r_i and s_i are chosen to be the smallest even degree which makes sense, i.e. $\deg(r_i)=4$ and $\deg(s_i)=2$.

E Computational Results of Different Approaches

The results shown here are for two-dimensional plane Couette flow with a length $L_{\text{dom}} = \frac{2\pi}{3.75}$ (where the height is unity), so that the principal wavenumber is $\alpha_1 = 3.75$. This is very close to the critical wavenumber obtained via the energy stability limit, and the energy stability limit in this domain is $\text{Re}_E = 177.3$.

The idea is to compare the different methods described (at least those that were implemented). With this in mind, we first show results at $\text{Re} = 179.5$ which is above the energy stability limit. We do this using six eigenmodes: the eigenmodes corresponding to the two largest eigenvalues at $\alpha_0 = 0$ and the same with $\alpha_1 = 3.75$. There are six, because for each eigenvalue associated to $\alpha \neq 0$ there are two eigenmodes: horizontal translation invariance requires choosing two linearly independent eigenmodes and we choose them to be $\frac{\pi}{2}$ out of phase to ensure orthogonality. We refer to this set of six eigenmodes as "Modes I." Additionally, "Modes II" has all the modes in "Modes I" with two additional associated to the highest eigenvalue of $\alpha_2 = 7.5$, for a total of eight modes. All experiments were performed with $\deg(V) = 4$. A value of $\delta_i = 1$ was used at every point where the "Peter-Paul" inequality was utilized. Lastly, in the Bounding method (§D) it was used that $\bar{\varepsilon} = 10^{-8}$ and $\bar{\bar{\varepsilon}} = 10^{-4}$. To compare the results and computational performance of each algorithm, it is useful to show the final value of ε, as this gives a rough idea of how effective the algorithm is (the higher the value of ε the better), and we also are showing the time inside the SDP solver to give a rough estimate of the computational costs, which are typically high. Having said that, the total computation time consists of: (i) the time taken to solve the eigenvalue problem and compute the relevant tensors ($\mathbf{L}$, $\boldsymbol{\mathcal{N}}$, $\mathbf{R}_i$ and C_i in (2.6) and (A.3)), (ii) the time to setup the SDP (i.e., computing the dynamical system itself along with all the SDP constraints), (iii) the preprocessing time to find symmetries and parse to the appropriate format compatible with a given SDP solver, and (iv) the SDP solver time. The results are shown in Table 1.

SDP	Re = 179.5			
	Modes I		Modes II	
	ε	SDP solver time (s)	ε	SDP solver time (s)
Original 1	2.54×10^{-6}	48	–	–
Original 2	0.0030	48	–	–
Method 1	0.0175	210	0.1495	3462
Method 2	0.0170	235	0.1512	8931
Bounding	–	339	–	5195

Table 1: Performance of the methods at a fixed Re = 179.5 and for two different sets of modes.

In terms of ε, the two methods that seem to be the most robust are Method 1 and Method 2, but for some reason, the computational cost of Method 2 is actually higher and does not scale well as more modes are added, so Method 1 is preferred. Also, Original 1 and Original 2 behave very similarly, as expected, but Original 1 produces a value of ε which is safer to trust. Unfortunately, large memory requirements did not allow for simulations to be completed with Modes II via Original 1 and Original 2. Thus, these methods do not scale well as more modes are added, and to overcome this, probably an alternative implementation that does not use auxiliary variables is required (see Section A.3 for more details). Lastly, since the Bounding method is based on Method 1, it was to be expected that the cost was higher than that of Method 1, which indeed is the case.

Next, the results are shown for both sets of modes, but this time showing the highest Reynolds number at $L_{\text{dom}} = \frac{2\pi}{3.75}$ for which each method was found to be stable. These are presented in Table 2.

SDP	Best Re attaining global stability			
	Modes I		Modes II	
	Re	ε	Re	ε
Original 1	179.5	2.54×10^{-6}	–	–
Original 2	179.5	0.0030	–	–
Method 1	194	1.60×10^{-4}	230	1.07×10^{-5}
Method 2	195	2.97×10^{-5}	228	9.16×10^{-5}
Bounding	190	–	214	–

Table 2: Best Re resulting in global stability for each method and for two different sets of modes.

Clearly, Method 1 seems to be the best in terms of obtaining the largest value of Re (combined with a better computational performance over Method 2). Method 2 and the Bounding method perform quite well but are more expensive than Method 1. Meanwhile the Original 1 and Original 2 methods perform almost the same and do not go very high. When adding more modes, the question was if higher Re could be attained or if the bounds would degenerate to the point where it was not viable to add more modes. Thankfully, it seems, up to the precision of the solvers, that for Method 1 and Method 2 the performance does improve (Method 2 is much more expensive). In any case, the results are satisfactory, since the highest value one could hope for with these sets of modes (i.e. where the truncated system becomes unstable with both Modes I and Modes II) was Re = 266, and Method 1 resulted in Re = 230, which is not very far. Experimenting with different δ_i might improve the results of Method 2 even further. In the future, Method 3 could provide significant advantages in terms of speed, and is certainly worth looking into. On the other hand, the method using much better bounds for $\mathbf{M} \cdot \mathbf{\Theta_{ab}}$ seems very expensive but it could be useful to try out eventually, as it may improve the global stability results.

Note that all results reported here were implemented using MATLAB via YALMIP. The SDP solver used was MOSEK. To cement the confidence in the results, they could potentially be verified with multiple-precision (using the solver SDPA-GMP) or with interval arithmetic (using the solver VSDP which relies on INTLAB), but such verification has not been done.

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