Cooling via Baroclinic Acoustic Streaming

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Introduction

As most waves, sound waves have the faculty to drive steady Eulerian flows. This effect, called acoustic streaming, can be sensed as a consequence of the conservation of momentum. Indeed, consider a progressive acoustic wave subject to an attenuation mechanism, *e.g.* viscosity or thermal conductivity: whereas the energy of the wave shall be mostly transferred to internal energy (and result in a local heating), a mean flow must be generated in order to conserve momentum. Such non-zero Eulerian mean motions play a crucial role in mass and heat transport. This has been recognized for water waves by Longuet-Higgins [1], where it is of the same order as the Stokes drift (the difference between Lagrangian and Eulerian mean velocities). For sound waves, acoustic streaming is by far the main transport mechanism [2]. Given that powerful ultrasonic sources are nowadays of common use, acoustic streaming can be considered a simple and low-cost way to enhance heat transfer [3]. In this study, we investigate the effect of acoustic waves on a strong, stably stratified medium, namely a fluid in between two parallel plates of very different temperatures.

In the absence of thermal driving, the streaming flow generated by plane standing waves has been worked out by Rayleigh [4] in the limit $H_* \gg \delta_{BL}$, where H_* is the width of the system and δ_{BL} is the width of the boundary layers (for a more general study, see [5]). For a plane wave of the form $U_* \cos(k_* \tilde{x})\vec{e}_x$, it consists of two series of vortices located symmetrically about the median plane, of typical velocity $3U^2/(16a_*)$, where a_* is the speed of sound and k_* the wavenumber. Note that, even though acoustic streaming results in this case from viscous dissipation in the Stokes boundary layers, the mean flow does not depend on the viscosity, and therefore does not vanish in the limit of infinitely small dissipation. Assuming that both the mean flow and the acoustic waves are not affected by heat, *i.e.* that the density does not depend on temperature, the additional heat flux associated with this streaming flow has been computed for small aspect ratio [6]. Experiments have confirmed that acoustic streaming enhances heat transfers [7]. Direct numerical simulations have also been performed for relatively small aspect ratio, in the absence of gravity [8, 9], and show that moderate thermal driving results in a vertical merging of these stack cells and in an increase of the velocity of the streaming flow. This solution strongly contrasts with the one theoretically considered for the computation of the heat flux [6].

This discrepancy between full DNS and previous theoretical studies results from the assumption that the mean flow is driven by acoustic streaming taking place in the viscous boundary layers. However, it has been recently recognized that in the presence of a stratification, the dominant driving force takes place in the bulk and results from baroclinic production of vorticity of acoustic waves [10]. Acknowledging this mechanism as the leading one, Chini *et al.* were able to obtain the correct order of magnitude for the mean velocity in a strongly stratified system driven by acoustic streaming (a stabilized HIV lamp [11]), whereas usual Rayleigh streaming would lie two orders of magnitude below. Along the same lines of these authors, we demonstrate in this report that a similar description can be applied to a gas between two plates subjected to a intense temperature difference.

We perform a multi-scale analysis of this problem and obtain governing equations for the mean flow and for the acoustic waves. This reduced model emphasizes the complex dynamics of the system, as waves and streaming flow present a two-way coupling: acoustic waves drive a mean flow, that in return modifies the density field and thus affects the wave field. We derive an approximate solution in some range of parameters, that provides an accurate model for the previously mentioned direct numerical simulation of the full system. We also present numerical simulations of this reduced set of equations, that can describe regimes with strong coupling.

This report is organized as follows: we first review the basic mechanisms of acoustic streaming, then describe the system and the multiple scale analysis. In section 3, we show that the acoustic wave field can be obtained as the solution of a one-dimensional eigenvalue problem, and describe how its amplitude evolves. In section 4, we then compute an approximate solution of this system and evaluate the associated heat flux and efficiency. This is compared to previous direct numerical simulations of the full problem. Finally, in section 6, we present the results of numerical simulations, then draw our conclusion.

1 Basic Mechanisms of Acoustic Streaming

1.1 The role of vorticity

In order to emphasize the role of stratification in acoustic streaming, we first review some very basic facts about nonlinearities in acoustics. In all the following, acoustic fields will be assumed to be of small amplitudes compared to the speed of sound a_*. As a result, nonlinear terms in the Navier-Stokes equation do not much affect the waves to a first approximation. Indeed, this governing equation reads, for a Newtonian compressible fluid without second viscosity,

$$\rho\left[\partial_t\vec{v}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right]=-\vec{\nabla}P+\vec{f}+\eta\left(\triangle\vec{v}+\frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right), \tag{1}$$

and, with ω the angular frequency of the wave, k its wavenumber and U the wave amplitude,

$$\left|\frac{\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}}{\partial_t\vec{v}}\right|\sim\frac{U^2k}{\omega U}\sim\frac{U}{a_*}\ll 1. \tag{2}$$

Therefore, nonlinearities appear as a small correction for the waves dynamics. However, it would be unfortunate to disregard their consequences based on this fact since, although small, such effects may be cumulative and affect the long-time evolution of the wave field. Given the time-dependence of the waves, the nonlinear term will contain high frequencies: this results in the apparition of harmonics in the signal, and may also lead to shock waves or transfers of energy between wave trains (acoustic waves undergo three-waves or more interactions). Such high frequencies can still be regarded as waves, and will not be further discussed.

In the present work, we are interested in the low-frequencies that may be driven by this nonlinear term. For this purpose, it is useful to cast it as

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \vec{\nabla}\left(\frac{\vec{v}^2}{2}\right) + \left(\vec{\nabla}\times\vec{v}\right)\times\vec{v}. \tag{3}$$

This evidences that, although constant terms shall always result from quadratic nonlinearities of an oscillating field, part of them are balanced by a pressure variation[1]. Therefore, in order to drive a mean flow, acoustic waves must have some vorticity.

It is then natural to wonder the conditions necessary for an acoustic field to acquire vorticity. This has been so far mostly discussed for fluids with uniform density background. In this case vorticity is generated in an irrotational flow by viscosity [12, 13], although external forces or moving boundary can also be considered. We shall see that another strong source of vorticity resides in an inhomogeneous background density ρ_0. This can be evidenced by taking the curl of the linear Euler equation, that describes inviscid and linear acoustics,

$$\vec{\nabla}\times\left(\rho_0\partial_t\vec{v} = -\vec{\nabla}P\right) \Longrightarrow \partial_t\left(\vec{\nabla}\times\vec{v}\right) = \frac{(\vec{\nabla}\rho)\times(\vec{\nabla}P)}{\rho_0^2}. \tag{4}$$

The left-hand side of this equation is the so-called "baroclinic contribution", and is non-zero when isobars and isopycnals differ.

1.2 Acoustic streaming in a horizontal cavity

As a first approach of acoustic streaming, we review the theory in a channel with an uniform density background, mostly done by Rayleigh [4]. It results from vorticity being generated in the thin boundary layers, contrary to "quartz wind", in which viscosity acts in the bulk during the propagation of a wave train. We consider two parallel boundaries separated by a height H_*, of same temperature, and describe the steady state associated with a plane standing wave in the x direction, of wavenumber k_*. In all the following, tildes and stars refer to dimensional quantities, boldfaces to vectors, bars to time-averaged quantities and primes to oscillating fields. The setup is sketched in Fig. 1, and we assume that a steady-state is reached. To derive the streaming flow, we proceed as follows:

1. We assume that an operator drives an acoustic wave along the x direction in the bulk, then compute the corrections caused by the presence of boundary layers.
2. In the boundary layers, we define and compute the "Reynolds stress", *i.e.* the mean force density that acts on the streaming flow.
3. We balance this force with viscosity in the boundary layers, and show that it results in an effective slip velocity for the streaming flow.

[1] Because acoustic waves are compressible flows, one should not forget about the term $(\rho-\rho_0)\partial_t\vec{v}$, where ρ_0 is a background density field that may, however, often be expressed as a gradient.

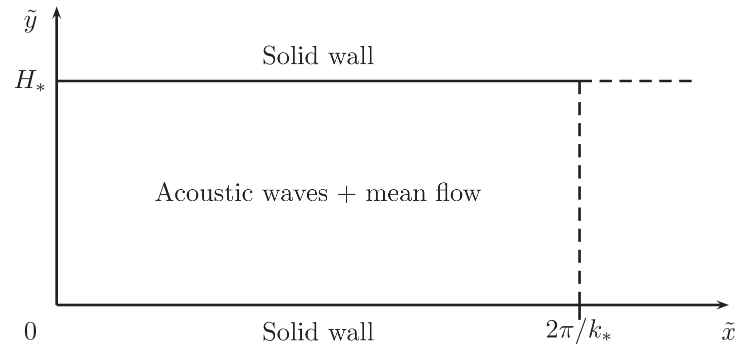

Figure 1: Rayleigh problem of acoustic streaming

Quantity	Expansion or scaling	Parameters involved
x velocity $\tilde{u}(\tilde{x},\tilde{y},\tilde{t})$	$\epsilon a_* \left(u_1(x,y,t) + \epsilon u_2(x,y,t) + \dots\right)$	The speed of sound a_*
y velocity $\tilde{v}(\tilde{x},\tilde{y},\tilde{t})$	$\epsilon^2 a_* \left(v_1(x,y,t) + \epsilon v_2(x,y,t) + \dots\right)$	The speed of sound a_*
Density $\tilde{\rho}(\tilde{x},\tilde{y},\tilde{t})$	$\epsilon\rho_* \left(1 + \epsilon\rho_1(x,y,t) + \dots\right)$	The background density ρ_*
Pressure $\tilde{p}(\tilde{x},\tilde{y},\tilde{t})$	$p_* + \rho_* a_*^2 \left(\epsilon p_1(x,y,t) + \dots\right)$	The background pressure p_*
$\tilde{t}$	$\omega_*^{-1} t$	The angular frequency ω_*
$\tilde{x}$	$k_*^{-1} x$	The wavenumber $k_* = \omega_*/a_*$
$\tilde{y}$	δy, where $\delta_{BL} = \sqrt{2\nu/\omega_*}$	The B.L. thickness δ_{BL}
Small parameter ϵ	$\epsilon = k_*\delta$	ϵ is dimensionless and small

Table 1: Scaling of the variables for the Rayleigh streaming

4. We compute the bulk flow by balancing this driving with viscosity, and compare the result to numerical simulations that include inertia.

1.2.1 Effect of the boundary layers on the acoustic field

The presence of solid boundaries in a fluid imposes no-slip boundary conditions, that can not be handled by potential flows. It thus generates both vorticity and strong velocity gradients, that are often the dominant damping mechanism (a typical example being sloshing). Quite surprisingly, the wave field actually undergoes changes *everywhere*, even far from the boundary layer. Here we derive the acoustic wave field at the leading order in the bottom boundary layer, as well as the small correction that affects the bulk flow.

In this problem, we have several small dimensionless parameters. The first one, previously mentioned, is the ratio of the wave amplitude to the speed of sound. Other ones compare the boundary layer thickness $\delta_{BL} = \sqrt{2\nu/\omega}$, of the order of a few microns for ultrasounds in air, to the acoustic wavelength and to the width of the system. For simplicity, these numbers are chosen equal and small, which results in the expansions reported in Table 1. This problem has four governing equations:

1. The continuity equation,

$$\partial_{\tilde{t}}\tilde{\rho} + \partial_{\tilde{x}}(\tilde{\rho}\tilde{u}) + \partial_{\tilde{y}}(\tilde{\rho}\tilde{v}) = 0, \tag{5}$$

that reads with our scalings and at the leading order

$$\partial_t \rho_1 + \partial_x u_1 + \partial_y v_1 = 0. \tag{6}$$

2. The equation of state for an isentropic evolution,

$$\beta_s = \frac{1}{\tilde{\rho}} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{p}} \right)_S , \tag{7}$$

with $a_* = 1/\sqrt{\beta_s \rho_*}$. At the leading order, it yields $p_1 = \rho_1$.

3. The Navier-Stokes equation along the y direction,

$$\tilde{\rho} \left(\partial_{\tilde{t}} \tilde{v} + \tilde{u} \partial_{\tilde{x}} \tilde{v} + \tilde{v} \partial_{\tilde{y}} \tilde{v} \right) = -\partial_{\tilde{y}} \tilde{p} + \rho_* \nu \left(\partial_{\tilde{x}\tilde{x}} \tilde{v} + \partial_{\tilde{y}\tilde{y}} \tilde{v} + \frac{1}{3} \partial_{\tilde{y}} (\partial_{\tilde{x}} \tilde{u} + \partial_{\tilde{y}} \tilde{v}) \right) , \tag{8}$$

that reduces at the leading order to

$$\partial_y p_1 = 0. \tag{9}$$

4. The Navier-Stokes equation along the x direction,

$$\tilde{\rho} \left(\partial_{\tilde{t}} \tilde{u} + \tilde{u} \partial_{\tilde{x}} \tilde{u} + \tilde{v} \partial_{\tilde{y}} \tilde{u} \right) = -\partial_{\tilde{x}} \tilde{p} + \rho_* \nu \left(\partial_{\tilde{x}\tilde{x}} \tilde{u} + \partial_{\tilde{y}\tilde{y}} \tilde{u} + \frac{1}{3} \partial_{\tilde{y}} (\partial_{\tilde{x}} \tilde{u} + \partial_{\tilde{y}} \tilde{v}) \right) , \tag{10}$$

that is at order $O(\epsilon)$

$$\partial_t u_1 = -\partial_x p_1 + \frac{\partial_{yy} u_1}{2}. \tag{11}$$

This set of equations has to be solved with a no-slip boundary condition at $y = 0$ and with the far-field assumed to be an acoustic standing wave of the form

$$u(x, y = \infty, t) = \cos(x) \cos(t). \tag{12}$$

This boundary condition (12) with (9) and (11) gives the order one pressure,

$$p_1(x, y, t) = \sin(x) \sin(t). \tag{13}$$

We can then get a close equation for u_1,

$$\partial_t u_1 = -\cos(x) \sin(t) + \frac{\partial_{yy} u_1}{2}, \tag{14}$$

that describes a Stokes boundary layer and is straightforward to solve,

$$u_1(x, y, t) = \cos(x) \left[\cos(t) \left(1 - \cos(y) e^{-y} \right) + \sin(t) \sin(y) e^{-y} \right] . \tag{15}$$

Now that we know both ρ_1 (from p_1 and the equation of state) and u_1, the continuity equation (6) with the boundary condition $v(x, y = 0, t) = 0$ result in

$$v_1 = \frac{\sin(x)}{2} \left[\cos(t) \left(-1 - \sin(y) e^{-y} + \cos(y) e^{-y} \right) + \sin(t) \left(1 - \sin(y) e^{-y} - \cos(y) e^{-y} \right) \right] . \tag{16}$$

Note that v_1 does not vanish in the limit $y \to \infty$, *i.e.* that the effect of the solid boundary is not restricted to the boundary layer. However, v is scaled as a small quantity compared to u, so that this velocity field remains a correction.

1.2.2 Definition and computation of the Reynolds stress

The streaming flow is a second order quantity, and we therefore need to consider (10) at order $O(\epsilon^2)$,

$$\partial_t u_2 + \rho_1 \partial_t u_1 + u_1 \partial_x u_1 + v_1 \partial_y u_1 = -\partial_x p_2 + \frac{\partial_{yy} u_2}{2} + \frac{1}{3}\partial_y \left(\partial_x u_1 + \partial_y v_1\right). \qquad (17)$$

The left hand side can be modified with the continuity equation (6),

$$\partial_t u_2 + \partial_t(\rho_1 u_1) = -\partial_x p_2 - \partial_x u_1^2 - \partial_y(u_1 v_1) + \frac{\partial_{yy} u_2}{2} + \frac{1}{3}\partial_y \left(\partial_x u_1 + \partial_y v_1\right). \qquad (18)$$

We then take the time-average of this equation,

$$0 = -\partial_x \bar{p}_2 - \overline{\partial_x u_1^2} - \overline{\partial_y(u_1 v_1)} + \frac{\partial_{yy}\bar{u}_2}{2}. \qquad (19)$$

We evidence on this simple system a general feature of acoustic streaming that consists of an effective force on the second order mean velocity field coming from inertial leading order terms. Generally speaking, this force $\mathbf{F} = F_j \mathbf{e}_j$ can be written as the divergence of the Reynolds stress, and is

$$\boxed{F_j = -\frac{\partial(\overline{\rho u_i u_j})}{\partial x_i}} \qquad (20)$$

where the repeated suffix i is summed over one to three. With (15) and (16), we obtain

$$F_x = \frac{\sin(2x)}{4}\left(2 + e^{-2y} + \sin(y)e^{-y} - 3\cos(y)e^{-y}\right). \qquad (21)$$

Whereas the constant term along the y direction can be handled by the pressure field $\bar{p}_2$, the other ones cannot.

1.2.3 Effective slip velocity

Most of the Reynolds stress divergence has to be balanced with viscosity, *i.e.* with the term $\partial_{yy}\bar{u}_2$, that reads

$$\partial_{yy}\bar{u}_2 = -\frac{\sin(2x)}{2}\left(\sin(y)e^{-y} - 3\cos(y)e^{-y} + e^{-2y}\right). \qquad (22)$$

With the boundary conditions $(\partial_y \bar{u}_2)(x, y = \infty, t) = 0$ and $\bar{u}_2(x, y = 0, t) = 0$, it provides

$$\bar{u}_2(x, y, t) = \frac{\sin(2x)}{8} e^{-y}\left(-3\sin(t) - \cos(y) + 2\sinh(y) + \cosh(y)\right). \qquad (23)$$

In particular, *the mean second order flow does not vanish far away from the boundary layer*, where it takes the value

$$\boxed{\bar{u}_2(x, y = \infty, t) = \frac{3\sin(2x)}{8}} \qquad (24)$$

This limit velocity is, with matched asymptotic expansion, a boundary condition for the mean flow. In particular, we emphasize that this quantity does not depend on the value of the viscosity, and is a small fraction of the acoustic wave amplitude. More precisely, if we denote by $U_* = \epsilon a_*$ the dimensional amplitude of the acoustic wave,

$$\bar{\tilde{u}}(\tilde{x}, \tilde{y} = 0, \tilde{t}) = \frac{3U_*^2}{8a_*}\sin(2k_*\tilde{x}). \qquad (25)$$

1.2.4 Mean flow in the bulk

If the driving imposed by acoustic streaming is balanced by viscosity (this regime being called "Rayleigh streaming"), we have to solve in the entire domain

$$\mathbf{0} = -\nabla\tilde{p} + \eta\triangle\bar{\tilde{\mathbf{u}}}_2, \qquad \bar{\tilde{u}}(\tilde{x}, \tilde{y} = 0 \text{ or } H_*, \tilde{t}) = \frac{3U_*^2}{8a_*}\sin(2k_*\tilde{x}). \tag{26}$$

The steady-state is found with the use of a stream function ψ ($\bar{\tilde{u}}_2 = \partial_{\tilde{y}}\psi$, $\bar{\tilde{v}}_2 = -\partial_{\tilde{x}}\psi$), that has to be a solution of $\nabla^4\psi = 0$. Note that

$$\sinh(ny)\sin(nx) \qquad \text{and} \qquad y\cosh(ny)\sin(nx) \tag{27}$$

are solutions of $\nabla^4\psi = 0$, so we can look for ψ of the form

$$\psi(\tilde{x}, \tilde{y}) = [A\sinh(n(\tilde{y} - H_*/2)) + Bn(\tilde{y} - H_*/2)\cosh(n(\tilde{y} - H_*/2))]\sin(n\tilde{x}). \tag{28}$$

We still have to enforce the boundary conditions. Canceling $\tilde{v}$ at the solid boundaries gives $B = -A(nH_*/2)^{-1}\tanh(nH_*/2)$. The effective slip condition fixes $n = 2\tilde{k}$ and B, so that

$$(\tilde{x}, \tilde{y}) = A\Psi(y)\sin(2\tilde{k}\tilde{x}), \tag{29}$$

with

$$A = \left(\frac{3U_*^2}{16A_*k_*}\right) \times \left(\operatorname{sech}(\tilde{k}H_*) - \sinh(\tilde{k}H_*)/(\tilde{k}H_*)\right)^{-1} \tag{30}$$

and

$$\Psi(y) = \sinh\left(2\tilde{k}(\tilde{y} - \frac{H_*}{2})\right) - \frac{\tanh(\tilde{k}H_*)}{\tilde{k}H_*}\left(2\tilde{k}(\tilde{y} - \frac{H_*}{2})\right)\cosh\left(2\tilde{k}(\tilde{y} - \frac{H_*}{2})\right). \tag{31}$$

This solution describes a set four vortices per acoustic wavelength, two in the horizontal direction, and two in the vertical one. Their energies are localized at a distance $\sim \tilde{k}^{-1}$ of the boundaries, so that the streaming velocity at the center of the cell becomes very weak if the aspect ratio $\tilde{k}H_*$ is large, see Fig. 2 . This is the reason why this regime of acoustic streaming is usually described in the limit $\delta \ll H \ll k_*^{-1}$, where the stream function then becomes

$$\psi(\tilde{x}, \tilde{y}) \simeq \frac{3U_*^2H_*}{4a_*}\left[\left(\frac{\tilde{y}}{H_*} - \frac{1}{2}\right)^3 - \frac{1}{4}\left(\frac{\tilde{y}}{H_*} - \frac{1}{2}\right)\right]\sin(2k_*\tilde{x}), \tag{32}$$

and the velocities are

$$\bar{\tilde{u}}_2(\tilde{x}, \tilde{y}) \simeq -\frac{3U_*^2}{4a_*}\left[\frac{1}{4} - 3\left(\frac{\tilde{y}}{H_*} - \frac{1}{2}\right)^2\right]\sin(2k_*\tilde{x}), \tag{33}$$

$$\bar{\tilde{v}}_2(\tilde{x}, \tilde{y}) \simeq -\frac{3U_*^2k_*H_*}{2a_*}\left[\left(\frac{\tilde{y}}{H_*} - \frac{1}{2}\right)^3 - \frac{1}{4}\left(\frac{\tilde{y}}{H_*} - \frac{1}{2}\right)\right]\cos(2k_*\tilde{x}). \tag{34}$$

A typical velocity induced in this system if often defined as the x velocity in the mid-plane, that is $3U_*^2/(16a_*)$.

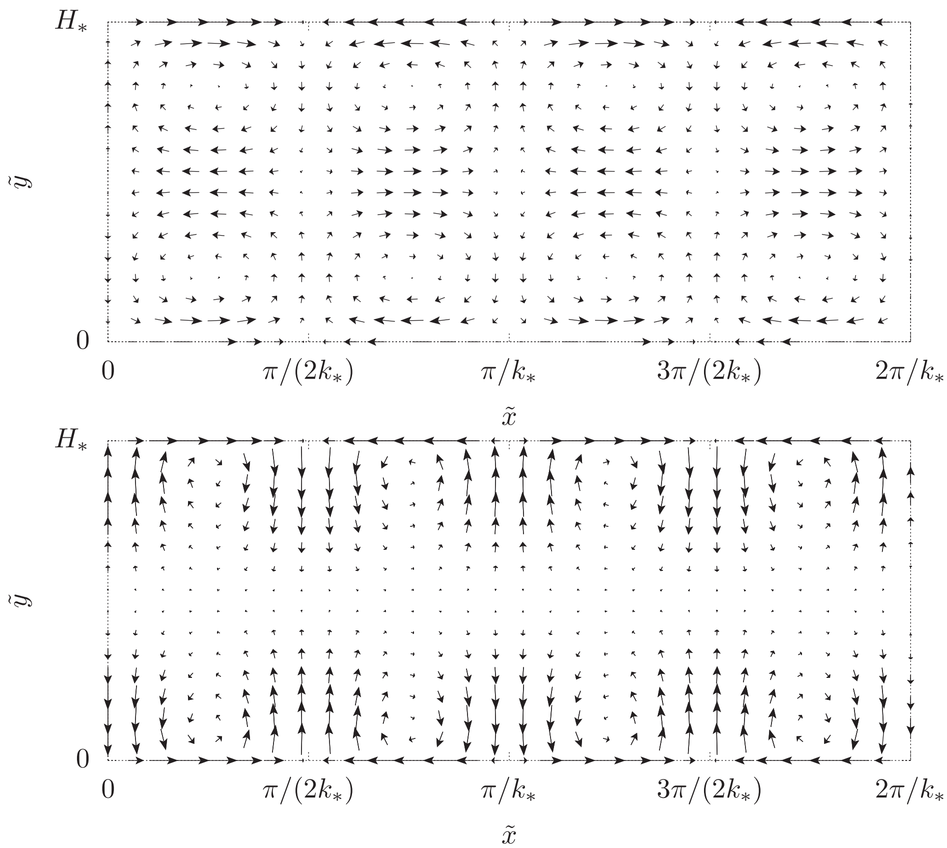

Figure 2: Rayleigh streaming for aspect ratios $k_*H_* = 0.5$ (top) and $k_*H_* = 5$ (bottom).

1.2.5 Effect of inertia ("Stuart streaming" or "Eckart streaming")

Up to now, we have considered that the flow in the bulk is fully balanced by viscosity, which may not always be valid. In order to discuss this assumption, we have to evaluate the streaming Reynolds number R_s, defined by

$$R_s = \frac{U_*^2 \ell}{a_* \nu}, \tag{35}$$

where ℓ is the relevant length-scale. We thereafter consider large and small aspect ratios.

Large aspect ratios If the aspect ratio is large, then the system does not depend on H_* anymore and $\ell = k_*^{-1}$, so that

$$R_s = \left(\frac{U_*}{a_*}\right)^2 \times \left(\frac{a_*}{k_* \nu}\right). \tag{36}$$

For an ideal gas (and, generally speaking, for most gas), the order of magnitude of the kinematic viscosity is the product of the speed of sound to the mean free-path ℓ_p, so that

$$R_s \sim \left(\frac{U_*}{a_*}\right)^2 \times \left(\frac{\lambda_*}{\ell_p}\right), \tag{37}$$

where $\lambda_* = 2\pi/k_*$. Therefore, although the first term is small, the streaming Reynolds number can still be large (for air in usual conditions, $\ell_p \sim 10^{-7}$m). Inertia results in a jet-like flow where the velocities are concentrated in a second boundary layer, whose length $\delta_{BL,MF}$ lies between the one of the acoustic waves $\delta_{BL} \sim \sqrt{\nu/\omega}$ and the wavelength λ_* [14]. It can be estimated based on the velocity U_*^2/a_* and the characteristic length k_*^{-1},

$$\delta_{BL,MF} \sim \sqrt{\frac{\nu}{k_*^{-1} \times (U_*^2/a_*)}} \sim \left(\frac{a_*}{U_*}\right) \times \delta_{BL} \gg \delta_{BL} \tag{38}$$

To illustrate this, we report in Fig. 3 the velocity field obtained with a direct numerical simulation of the bulk flow with $R_s = 100$ and $k_*H_* = 5$. This has been obtained with Dedalus [15]. Compared to Fig. 2, this clearly evidences this jet-like structure.

Small aspect ratios Similarly, if the aspect ratio is small, we rather define a streaming Reynolds number based on H_*,

$$R_h = \frac{U_*^2 H_*}{a_* \nu} = R_s \times (k_* H_*), \tag{39}$$

where R_s is defined in (36). The jet-like structures are less pronounced (see Fig. 3), but the effect of inertia can still be observed for large values of R_h.

1.3 Main features of Rayleigh streaming

The main features of this streaming flow can be summarized as:

1. The amplitude of the streaming flow is quadratic in the amplitude of the acoustic wave (see, *e.g.*, (25)).
2. In this setup, it results in stacked vortices of energy localized within a distance k_*^{-1} from the walls.
3. The streaming flow close to the boundary layer is directed toward the velocity nodes of the acoustic wave, *i.e.* away from the pressure nodes.

2 Governing Equations for the Waves and the Mean Flow

2.1 Notations and dimensional equations

The problem we consider is sketched in Fig. 4 and consists of a thin layer of an ideal gas in between two horizontal boundaries that drive the system toward a stably stratified steady state. Compared to before, we add this temperature difference and our claim is that the

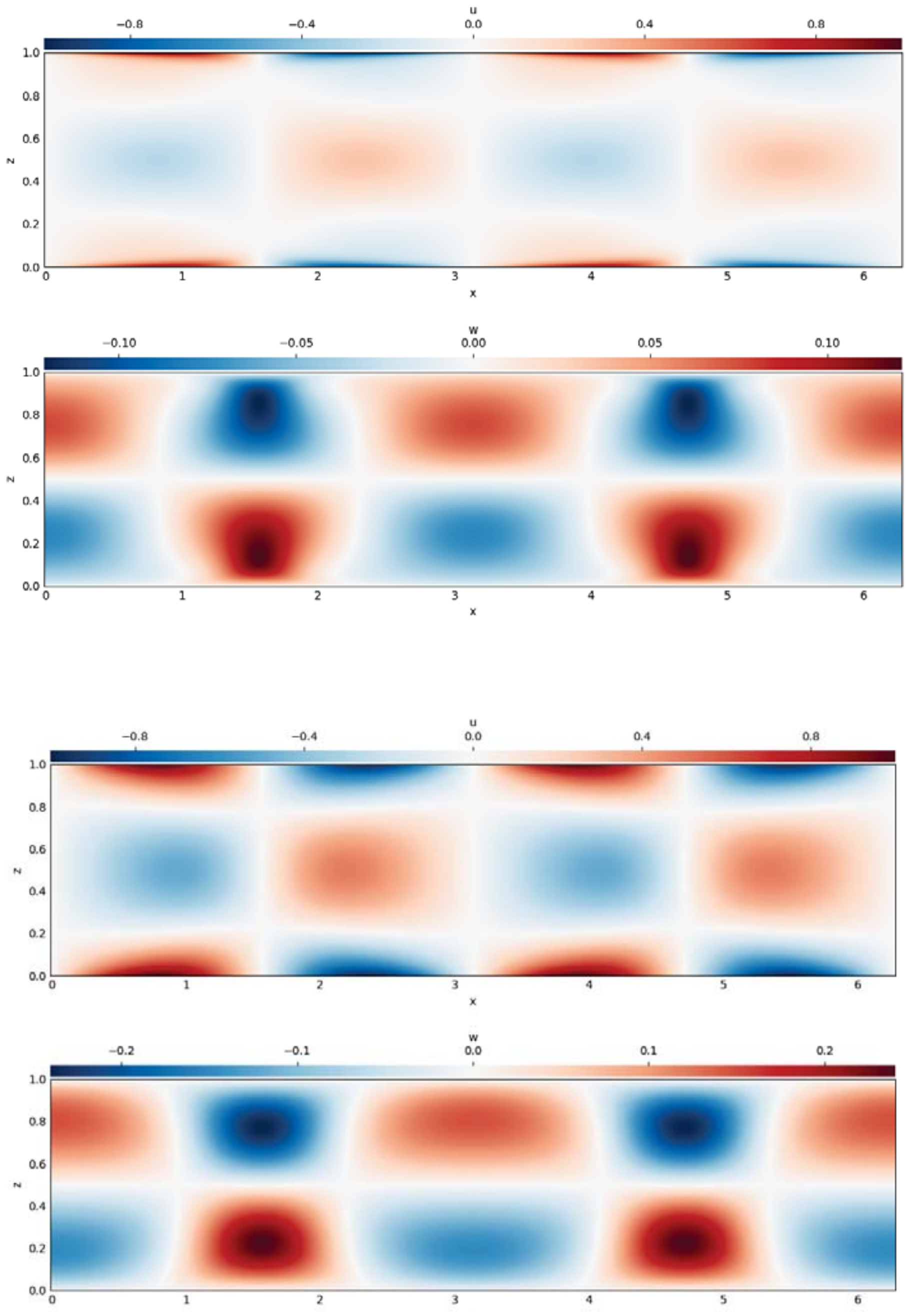

Figure 3: Rayleigh streaming for $k_* H_* = 5,\ R_s = 100$ (top two plots) and $k_* H_* = 0.5,\ R_h = 100$ (bottom two plots). u and w are the horizontal and vertical velocity fields, scaled with $3U_*^2/(8a_*)$, z is scaled with H_*, and x with k_*^{-1}.

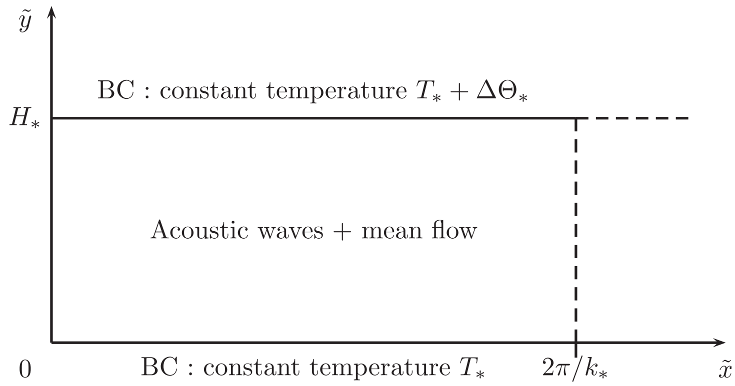

Figure 4: Schematic of the problem: acoustic waves interact with a mean flow in a thin layer of an ideal gas subject to a vertical thermal driving.

streaming flow will be driven by baroclinic vorticity in the bulk rather than viscosity in the boundary layers. For simplicity, we still consider two-dimensional flows and we neglect the effect of gravity[2]. The dimensional parameters and variables are defined in Table 2.

The kinematic boundary conditions are no-slip boundary conditions at $\tilde{y} = 0$ and $\tilde{y} = H_*$, and periodicity in the $\tilde{x}$ direction of period $2\pi/k_*$. Moreover, we fix $\tilde{u}(x = 0, y, t) = 0$ so that there is no exchange of mass between nearby cells. The thermal boundary conditions also consist of periodicity in $\tilde{x}$, and we require a constant temperature at the bottom, $\tilde{T}(\tilde{y} = 0) = T_*$, and at the top, $\tilde{T}(\tilde{y} = H_*) = T_* + \Delta\Theta_*$. We then define $\Gamma = \Delta\Theta_*/T_*$ as the dimensionless strength of this thermal driving. In the absence of flow, the dimensional temperature therefore reads

$$\tilde{T}_B(\tilde{y}) = T_* \left(1 + \Gamma \frac{\tilde{y}}{H_*}\right). \tag{40}$$

The dimensional equations are the same as in [10], excepted that we consider viscous heating. They are reported below, with $\nabla = (\partial_x, \partial_y)$.

$$\begin{cases} \tilde{\rho}\left[\partial_{\tilde{t}}\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{\mathbf{u}}\right] = -\tilde{\nabla}\tilde{p} + \mu\left[\tilde{\nabla}^2\tilde{\mathbf{u}} + \frac{1}{3}\tilde{\nabla}(\tilde{\nabla}\cdot\tilde{\mathbf{u}})\right], & (41)\\ \partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}}) = 0, & (42)\\ \tilde{\rho}c_v\left[\partial_{\tilde{t}}\tilde{T} + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{T}\right] = -\tilde{p}\left(\tilde{\nabla}\cdot\tilde{u}\right) + \kappa\tilde{\nabla}^2\tilde{T} - \Phi, & (43)\\ \tilde{p} = \tilde{\rho}R_s\tilde{T} & (44) \end{cases}$$

We consider the dynamic viscosity μ and the thermal conductivity κ to be independent of the temperature. The dissipation function Φ describes viscous heating and is

$$\Phi = 2\mu\left[(\partial_{\tilde{x}}\tilde{u})^2 + (\partial_{\tilde{y}}\tilde{v})^2 - \frac{(\partial_{\tilde{x}}\tilde{u} + (\partial_{\tilde{y}}\tilde{v})^2}{3}\right] + \mu\left(\partial_{\tilde{y}}\tilde{u} + \partial_{\tilde{x}}\tilde{v}\right)^2. \tag{45}$$

[2]Gravity would affect the equations at the leading order if the Richardson number $Ri = g_*/(k_* a_*^2)$ is scaled as $\epsilon^{3/2}$, which corresponds to a typical value of this number.

Notation	Definition
$(\tilde{x}, \tilde{y})$	(Horizontal, vertical) coordinate
H_*	Height of the channel
k_*	Horizontal wavenumber of the acoustic waves
κ	Thermal conductivity
$\tilde{\rho}$	Gas density
$\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$	Gas velocity
$\tilde{p}$	Pressure
$\tilde{T}$	Temperature
μ	Dynamic viscosity
R_s	Specific gas constant
(c_v, c_p)	Specific coefficient at constant (volume, pressure)
$a_* = \sqrt{(c_p/c_v)R_sT_*}$	Speed of sound
p_*	Equilibrium pressure at $\tilde{y} = 0$ if $\tilde{T}(\tilde{x}, \tilde{y}) = T_*$
U_*	Typical amplitude of the acoustic wave (velocity)

Table 2: Definitions of the dimensional parameters

Since viscous heating results from the viscous term in the Navier-Stokes equation, we have

$$\iint dxdy\Phi = \iint dxdy\ \mu\mathbf{u} \cdot \left[\tilde{\nabla}^2\tilde{\mathbf{u}} + \frac{1}{3}\tilde{\nabla}(\tilde{\nabla} \cdot \tilde{\mathbf{u}})\right]. \qquad (46)$$

2.2 Dimensionless equations

We now turn to dimensionless variables and equations. The scalings and dimensionless parameters used for this purpose are reported in Table 3 (for clarity, we define $\epsilon = S^{-1}$). Note that the aspect ratio has been chosen small. Even though the heat flux is not expected to be maximal in this regime, since the flow will mainly be along the x direction, this is motivated by the following facts. First, we expect the analysis of the acoustic wave-field to be simple in a domain thin compared to the wavelength. We shall see that it constrains the acoustic field to stay in the first mode along the vertical direction. Second, most theoretical and analytical studies have been performed in this regime, so that comparisons to previous works can be done. Third, at the leading order, the pressure gradient will be found orthogonal to the background density gradient, resulting in an important baroclinic contribution in the vorticity equation.

To illustrate these scalings, a setup in which $\epsilon = 10^{-3}$, and other parameters are perfectly scaled ($h = \Gamma = Re_s = Pe_s = 1$, $\gamma = 7/5$), would consist in a transducer sending a powerful audible sound ($f \simeq 600$Hz) in a strongly stratified long and thin layer ($H_* \simeq 3$ mm, $2\pi/k_* \simeq 60$ cm, $\Delta\Theta \simeq T_* \simeq 300$ K). Since ϵ is very small, h and $\Delta\Theta$ can be tuned so that a height of 3 cm and a temperature difference of 30 K can also be described by this system. The total temperature and pressure are written as a background profile plus a perturbation,

$$T(x, y, t) = T_B(y) + \Theta(x, y, t), \qquad P(x, y, t) = P_B(y) + \pi(x, y, t). \qquad (47)$$

We report below the dimensionless set of equations.

Variable	Scale	Parameter	Definition	Scaling
x	k_*^{-1}	Strouhal number S	a_*/U_*	$S = 1/\epsilon$
y	H_*	Aspect ratio δ	$k_* H_*$	$\delta = \sqrt{\epsilon} h$
t	$(a_* k_*)^{-1}$	Temperature gradient Γ	$\Delta\Theta_*/T_*$	$\Gamma = O(1)$
u	a_*	Reynolds number Re	$\rho_* U_*/(k_* \mu)$	$Re = Re_s/\epsilon$
v	$(k_* H_*) a_*$	Péclet number Pe	$\rho_* c_p U_*/(k_* \kappa)$	$Pe = Pe_s/\epsilon$
ρ	$\rho_* \equiv p_*/(R_s T_*)$	Specific heat ratio γ	c_p/c_v	$\gamma = O(1)$
T	T_*			
P	p_*			

Table 3: Definitions and scalings of the dimensionless parameters and variables

$$\rho\left[\partial_t u + (\mathbf{u}\cdot\nabla)\, u\right] = -\frac{1}{\gamma}\partial_x \pi + \frac{\epsilon^2}{Re_s}\left[\left(\partial_{xx} + \frac{1}{\epsilon h^2}\partial_{yy}\right) u + \frac{1}{3}\partial_x\left(\nabla\cdot\mathbf{u}\right)\right], \tag{48}$$

$$\rho\left[\partial_t v + (\mathbf{u}\cdot\nabla)\, v\right] = -\frac{1}{\epsilon\gamma h^2}\partial_y \pi + \frac{\epsilon^2}{Re_s}\left[\left(\partial_{xx} + \frac{1}{\epsilon h^2}\partial_{yy}\right) v + \frac{1}{3\epsilon h^2}\partial_y\left(\nabla\cdot\mathbf{u}\right)\right], \tag{49}$$

$$\partial_t\Theta + (\mathbf{u}\cdot\nabla)\,\Theta + v\frac{dT_B}{dy} = (1-\gamma)(T_B+\Theta)(\nabla\cdot\mathbf{u}) + \frac{\epsilon^2\gamma}{\rho Pe_s}\left(\partial_{xx} + \frac{1}{\epsilon h^2}\partial_{yy}\right)\Theta + O(\epsilon^3), \tag{50}$$

$$\partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \tag{51}$$

$$\rho = \frac{1+\pi}{T_B+\Theta}. \tag{52}$$

2.3 Expansion with respect to ϵ

To describe an acoustic field that evolves rapidly in time and whose properties depend on a slow modification of the density field, we introduce a slow time scale $T = \epsilon t$ and use the WKB approximation. Therefore, a function $f(x, y, t)$ becomes $f(x, y, \phi, T)$, where ϕ and T are independent variables. ϕ stands for the rapidly evolving phase, and may be written as

$$\phi(t) = \frac{\Phi(T)}{\epsilon}, \tag{53}$$

where $d\Phi/dT$ is of order one. We define the instantaneous angular frequency by

$$\omega(T) = \frac{d\phi}{dt} = \frac{d\Phi}{dT}. \tag{54}$$

In this framework, the time-derivative of f reads

$$\partial_t f \to \omega\partial_\phi f + \epsilon\partial_T f. \tag{55}$$

The fast time average of a function f is

$$\bar{f}(x,y,T) = \frac{1}{2n\pi}\int_{\phi}^{\phi+2n\pi} f(x,y,s,T)ds, \tag{56}$$

for sufficiently large positive integer n, so that any function can be split according to

$$f(x,y,\phi,T) = \bar{f}(x,y,T) + f'(x,y,\phi,T), \quad \bar{f}' = 0. \tag{57}$$

We then express all the fields as series of ϵ:

- $(u,v,\pi) = \epsilon(u_1,v_1,\pi_1) + \epsilon^2(u_2,v_2,\pi_2) + \dots$
- $(\Theta,\rho) = (\Theta_0,\rho_0) + \epsilon(\Theta_1,\rho_1) + \dots$
- $\Phi = \Phi_0 + \epsilon\Phi_1 + \dots$, so that $\omega = \omega_0 + \epsilon\omega_1 + \dots$

The derivation of governing equations for the streaming flow at the leading order can be found in [10], and are reproduced below,

$$\begin{cases} \bar{\rho}_0\left(\partial_T\bar{u}_1 + \bar{u}_1\partial_x\bar{u}_1 + \bar{v}_1\partial_y\bar{u}_1\right) = -\dfrac{\partial_x\bar{\pi}_2}{\gamma} - \partial_x\left(\bar{\rho}_0\overline{u_1'^2}\right) - \partial_y\left(\bar{\rho}_0\overline{u_1'v_1'}\right) + \dfrac{\partial_{yy}\bar{u}_1}{Re_s h^2} & (58)\\ \partial_y\bar{\pi}_2 = 0 & (59)\\ \partial_T\bar{\rho}_0 + \partial_x(\bar{\rho}_0\bar{u}_1) + \partial_y(\bar{\rho}_0\bar{v}_1) = 0 & (60)\\ \partial_T\bar{\Theta}_0 + \bar{u}_1\partial_x\bar{\Theta}_0 + \bar{v}_1\partial_y\left(\bar{\Theta}_0 + T_B\right) = (1-\gamma)(\bar{\Theta}_0 + T_B)(\partial_x\bar{u}_1 + \partial_y\bar{v}_1) + \dfrac{\gamma\partial_{yy}\bar{\Theta}_0}{Pe_s h^2\bar{\rho}_0} & (61)\\ \bar{\rho}_0 = \dfrac{1}{\bar{\Theta}_0 + T_B} & (62) \end{cases}$$

Substituting the density $\bar{\rho}_0$ with (62), it also reads

$$\begin{cases} \dfrac{\partial_T\bar{u}_1 + \bar{u}_1\partial_x\bar{u}_1 + \bar{v}_1\partial_y\bar{u}_1}{\bar{\Theta}_0 + T_B} = -\dfrac{\partial_x\bar{\pi}_2}{\gamma} - \partial_x\left(\dfrac{\overline{u_1'^2}}{\bar{\Theta}_0 + T_B}\right) - \partial_y\left(\dfrac{\overline{u_1'v_1'}}{\bar{\Theta}_0 + T_B}\right) + \dfrac{\partial_{yy}\bar{u}_1}{Re_s h^2} & (63)\\ \partial_y\bar{\pi}_2 = 0 & (64)\\ \partial_x\bar{u}_1 + \partial_y\bar{v}_1 = \dfrac{\partial_{yy}\bar{\Theta}_0}{Pe_s h^2} & (65)\\ \partial_T\bar{\Theta}_0 + \bar{u}_1\partial_x\bar{\Theta}_0 + \bar{v}_1\partial_y\left(\bar{\Theta}_0 + T_B\right) = (\bar{\Theta}_0 + T_B)\dfrac{\partial_{yy}\bar{\Theta}_0}{Pe_s h^2} & (66)\\ \bar{\rho}_0 = \dfrac{1}{\bar{\Theta}_0 + T_B} & (67) \end{cases}$$

On the other hand, the equations describing the acoustic waves are

$$\begin{cases} \omega_0\bar{\rho}_0\partial_\phi u_1' + \dfrac{1}{\gamma}\partial_x\pi_1' = 0 & (68)\\ \partial_y\pi_1' = 0 & (69)\\ \omega_0\partial_\phi\rho_1' + \partial_x(\bar{\rho}_0 u_1') + \partial_y(\bar{\rho}_0 v_1') = 0 & (70)\\ \omega_0\partial_\phi\Theta_1' + u_1'\partial_x\bar{\Theta}_0 + v_1'\partial_y\left(\bar{\Theta}_0 + T_B\right) + (\gamma-1)(\bar{\Theta}_0 + T_B)(\partial_x u_1' + \partial_y v_1') = 0 & (71)\\ \pi_1' - \rho_1'(\bar{\Theta}_0 + T_B) - \bar{\rho}_0\Theta_1' = 0 & (72) \end{cases}$$

We obtain a close set of equations that describes the coupled dynamics of a wave field and a streaming flow. On a fast time-scale, the acoustic waves are not affected by attenuation mechanisms (viscosity and thermal diffusivity). They are expected to affect the waves on the slow time-scale, and this effect is not taken into account by this set of equations. We shall see later on that such effects, among which are damping and energy transfer with the mean flow, are part of a second order solvability condition.

2.4 Energy balance

2.4.1 Dimensional energy balance

Although not essential for the rest of the study, it it interesting to have a look at the energy balance. For this system, it is (see Appendix A for more details),

$$\boxed{\frac{\mathrm{d}E_c}{\mathrm{d}t} = \dot{Q}} \tag{73}$$

where $\dot{Q}$ is the total heat flux received by the gas, defined by

$$\dot{Q} = \kappa \int dx \left[(\partial_{\tilde{y}}\tilde{T})(\tilde{x}, \tilde{y} = 0, \tilde{t}) - (\partial_{\tilde{y}}\tilde{T})(\tilde{x}, \tilde{y} = H_*, \tilde{t})\right], \tag{74}$$

and E_c the kinetic energy, defined by

$$E_c = \iint dxdy \left(\frac{\tilde{\rho}\tilde{\mathbf{u}}^2}{2} + \frac{\tilde{p}}{\gamma - 1}\right). \tag{75}$$

In (75), we recognize the macroscopic kinetic energy density and the microscopic kinetic energy density of an ideal gas, *i.e.* the internal energy of an ideal gas[3].

2.4.2 Dimensionless energy balance expended with respect to ϵ

With dimensionless quantities, (73) becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \iint dxdy \left[\frac{p}{\gamma(\gamma-1)} + \frac{\rho}{2}\left(u + \sqrt{\epsilon} hv\right)^2\right] \\ &\qquad = \frac{\epsilon}{h^2 Pe_s(\gamma-1)} \int dx \left[(\partial_y T)(x, y = 0, t) - (\partial_y T)(x, y = 1, t)\right]. \end{aligned} \tag{76}$$

That gives, at order ϵ and with fast-time averaging,

$$0 = \int dx \left[(\partial_y \bar{\Theta}_0)(x, y = 0, t) - (\partial_y \bar{\Theta}_0)(x, y = 1, t)\right], \tag{77}$$

that can be also derived from the initial set of equations. This states that the instantaneous heat fluxes at the top and at the bottom are equal.

[3]Since we assume that C_v does not depend on T, $U(T) = C_v T = \frac{nRT}{\gamma-1} = \frac{PV}{\gamma-1}$ for an ideal gas.

At order ϵ, for the fast time scale, we get

$$\iint dxdy \left(\partial_\phi \pi_1'\right) = 0, \tag{78}$$

given that Θ_0 does not depend on the fast time. This equality also results from the study of the acoustic modes ($\pi_1' \propto g(x)$, see next section).

At order ϵ^2, with fast-time averaging,

$$\iint dxdy \left(\partial_T \bar{\pi}_1\right) = \frac{\gamma}{h^2 Pe_s} \int dx \left[(\partial_y \bar{\Theta}_1)(x, y=0, t) - (\partial_y \bar{\Theta}_1)(x, y=1, t)\right]. \tag{79}$$

This balances the internal energy variation of the gas with the instantaneous heat flux. Given that $\bar{\pi}_1(x, y, T) = \bar{\pi}_1(T)$, this equation gives access to $\bar{\pi}_1$, via

$$\frac{d\bar{\pi}_1}{dT} = \frac{\gamma}{2\pi h^2 Pe_s} \int dx \left[(\partial_y \bar{\Theta}_1)(x, y=0, t) - (\partial_y \bar{\Theta}_1)(x, y=1, t)\right]. \tag{80}$$

3 Properties of the Acoustic Waves

We now focus on the acoustic waves. Their evolution only depends on one slow variable, the first order density $\bar{\rho}_0$ ($\bar{\Theta}_0 + T_B = \bar{\rho}_0^{-1}$). This quantity is considered as a given function in this section. Therefore, the set of governing equations is linear, and we thereafter consider only one eigenvector. All the fields can then be expressed as

$$f_1'(x, y, \phi, T) = \frac{1}{2}\left(A(T)\hat{f}_1(x, y, T)e^{i\phi} + c.c.\right), \tag{81}$$

where f stands for any variable (u,v,ρ,π,Θ), $A(T)$ is a slowly evolving amplitude, and $\hat{f}_1$, a complex function, describes the geometry of the mode. For this decomposition to be unique, we need a normalization condition for $\hat{f}$, that we shall derive later on. The slow evolution of A cannot be obtained from the first order set of equations, and will be found as a solvability condition for the waves at the next order.

3.1 Geometry of the mode

3.1.1 Reduction to a single ode

Given the decomposition (81), the governing equations are

$$\begin{cases} i\omega_0 \bar{\rho}_0 \hat{u}_1 + \frac{1}{\gamma}\partial_x \hat{\pi}_1 = 0 & (82) \\ \partial_y \hat{\pi}_1 = 0 & (83) \\ i\omega_0 \hat{\rho}_1 + \partial_x(\bar{\rho}_0 \hat{u}_1) + \partial_y(\bar{\rho}_0 \hat{v}_1) = 0 & (84) \\ i\omega_0 \hat{\Theta}_1 + \hat{u}_1 \partial_x \bar{\Theta}_0 + \hat{v}_1 \partial_y \left(\bar{\Theta}_0 + T_B\right) = (1-\gamma)(\bar{\Theta}_0 + T_B)(\partial_x \hat{u}_1 + \partial_y \hat{v}_1) & (85) \\ \hat{\pi}_1 = \hat{\rho}_1(\bar{\Theta}_0 + T_B) + \bar{\rho}_0 \hat{\Theta}_1 & (86) \end{cases}$$

This system becomes, with $\bar{\Theta}_0(x,y) + T_B(y) = 1/\bar{\rho}_0$,

$$\begin{cases} i\omega_0\bar{\rho}_0\hat{u}_1 + \dfrac{1}{\gamma}\partial_x\hat{\pi}_1 = 0 & (87)\\ \partial_y\hat{\pi}_1 = 0 & (88)\\ \hat{\rho}_1 = \dfrac{i}{\omega_0}\left[\partial_x(\bar{\rho}_0\hat{u}_1) + \partial_y(\bar{\rho}_0\hat{v}_1)\right] & (89)\\ \hat{\Theta}_1 = \dfrac{i}{\omega_0}\left[\hat{u}_1\partial_x\bar{\rho}_0^{-1} + \hat{v}_1\partial_y\bar{\rho}_0^{-1} - \dfrac{1-\gamma}{\bar{\rho}_0}(\partial_x\hat{u}_1 + \partial_y\hat{v}_1)\right] & (90)\\ \hat{\pi}_1 = \dfrac{\hat{\rho}_1}{\bar{\rho}_0} + \bar{\rho}_0\hat{\Theta}_1 & (91)\end{cases}$$

Combining (89), (90) and (91), $\hat{\pi}_1$ can then be expressed as a function of $\hat{u}_1$ and $\hat{v}_1$,

$$\hat{\pi}_1 = \frac{i\gamma}{\omega_0}(\partial_x\hat{u}_1 + \partial_y\hat{v}_1). \tag{92}$$

Therefore, the initial set of equations reduces to two coupled partial differential equations,

$$\begin{cases} \partial_x(\partial_x\hat{u}_1 + \partial_y\hat{v}_1) = -\omega_0^2\bar{\rho}_0\hat{u}_1 & (93)\\ \partial_y(\partial_x\hat{u}_1 + \partial_y\hat{v}_1) = 0 & (94)\end{cases}$$

We can go further and obtain a single ordinary differential equation. To this end, we formally integrate a combination of these equations,

$$\partial_y(93) + \partial_x(94) \Longrightarrow \partial_y(\bar{\rho}\hat{u}_1) = 0 \Longrightarrow \hat{u}_1 = \frac{f(x)}{\bar{\rho}_0}, \tag{95}$$

where f is an unknown function of x only. (94) then becomes

$$\partial_x\left(\frac{f(x)}{\bar{\rho}_0}\right) + \partial_y\hat{v}_1 = g(x), \tag{96}$$

where g is another unknown function. (93) finally reads

$$g' = -\omega_0^2 f \Longrightarrow f(x) = -\frac{g'(x)}{\omega_0^2}. \tag{97}$$

Equations (95) and (96) provide expressions for $\hat{u}_1$ and $\hat{v}_1$ as a function of this unknown function g only (remember that the bottom boundary condition is $\hat{v}_1(x, y=0, T) = 0$):

$$\begin{cases} \hat{u}_1(x,y,T) = -\dfrac{g'(x)}{\omega_0^2\bar{\rho}_0(x,y)} & (98)\\ \hat{v}_1(x,y,T) = yg(x) + \partial_x\left(\dfrac{g'(x)}{\omega_0^2}\displaystyle\int_0^y \frac{dy}{\bar{\rho}_0}\right) & (99)\end{cases}$$

The upper boundary condition $\hat{v}_1(x, y=1, T) = 0$ provides a differential equation for g,

$$\boxed{g(x) = -\frac{d}{dx}\left(\frac{g'(x)\alpha(x)}{\omega_0^2}\right) \Longleftrightarrow g''(x) + \frac{\alpha'}{\alpha}g'(x) + \frac{\omega_0^2}{\alpha}g(x) = 0} \tag{100}$$

where $\alpha(x)$ is defined by

$$\alpha(x) = \int_0^1 \frac{dy}{\bar{\rho}_0(x,y)}. \tag{101}$$

The ability to describe a two-dimensional acoustic field with a single ordinary differential equation is a result of the thin layer approximation: in any container of aspect ratio of order unity, the frequencies of the modes have to be found through a two-dimensional eigenvalue problem. This provides a huge simplification to the analysis of these acoustic modes, and we thereafter characterize this function g.

3.1.2 General features of g

Real-valued function We impose $\hat{u}_1$ to be a real field. Equations (93) and (94) thus imply that $\hat{v}_1$ is also a real field, (92) that $\hat{\pi}_1$ is a pure imaginary one, and so on. Given that $\hat{u}_1$ and $\hat{v}_1$ are directly related to g (see (98) and (99)), we deduce that g is real-valued.

Orthogonality (100) is a second order differential equation in the "Sturm-Liouville form" (or "self-adjoint form"), and cannot be solved explicitly. Its mechanical equivalent is the motion of a mass attached to a spring of variable stiffness (note that $\alpha > 0$), and driven or damped by a linear friction force. We therefore expect that g is a function "that oscillates". This is confirmed by the following integral, computed with the 2π periodicity in x:

$$\int_0^{2\pi} g(x)dx = -\left[\frac{g'(x)\alpha(x)}{\omega_0^2}\right]_0^{2\pi} = 0. \tag{102}$$

Thus, there must exist one or more x_0 such that $g'(x_0) = 0$, which correspond to nodes for the x-velocity: $\hat{u}_1(x_0, y, T) = 0$. To enforce the zero mass exchange at $x = 0$, the boundary conditions of (100) must then be $g'(0) = g'(1) = 0$[4]. We can also show that the eigenvectors of this ode are orthogonal. Let (g_A, g_B) be two eigenvectors and (ω_A, ω_B) their angular eigenfrequencies,

$$(\omega_A^2 - \omega_B^2)\int_0^{2\pi} g_A(x)g_B(x)dx = \int_0^{2\pi} \left[(\omega_A^2 g_A)g_B - g_A(\omega_B^2 g_B)\right] dx \tag{103}$$

$$= \int_0^{2\pi} \left[g_A\frac{d}{dx}(g_B'\alpha) - g_B\frac{d}{dx}(g_A'\alpha)\right] dx \tag{104}$$

$$= \left[\alpha(g_A g_B' - g_B g_A')\right]_0^{2\pi} = 0 \tag{105}$$

This provides a scalar product on eigenvectors, and we therefore require them to be normalized, *i.e.*

$$\boxed{\int_0^{2\pi} g(x)^2 dx = 1} \tag{106}$$

[4] With (58), this implies that $\bar{u}(x = 0, t, y)$ is always zero.

Vertically averaged velocities We define $(\hat{U}_1, \hat{V}_1)$ as the vertically averaged x and y velocity:

$$\hat{V}_1(x) = \int_0^1 \hat{v}_1(x,y)dy = \frac{g(x)}{2} + \frac{d}{dx}\left(\frac{g'(x)}{\omega_0^2}\int_0^1 dy \int_0^y \frac{dy'}{\bar{\rho}_0(x,y',T)}\right) \tag{107}$$

$$= \frac{g(x)}{2} + \frac{d}{dx}\left(\frac{g'(x)}{2\omega_0^2}\int_0^1 dy \int_0^1 \frac{dy'}{\bar{\rho}_0(x,y',T)}\right) \tag{108}$$

$$= \frac{g(x)}{2} + \frac{d}{dx}\left(\frac{g'(x)\alpha(x)}{2\omega_0^2}\right) = \frac{g(x)}{2} - \frac{g(x)}{2} = 0. \tag{109}$$

Therefore, the acoustic field has a zero vertical mean. For the horizontal velocity, we get

$$\hat{U}_1(x) = \int_0^1 \hat{u}_1(x,y)dy = -\frac{g'(x)}{\omega_0^2}\int_0^1 \frac{dy}{\bar{\rho}_0(x,y)} = -\frac{g'(x)\alpha(x)}{\omega_0^2}. \tag{110}$$

Similarly, we can derive some kind of orthogonality condition for these mean x velocities. Let $(\hat{U}_A, \hat{U}_B)$ be two eigenvectors, associated with the angular eigenfrequencies (ω_A, ω_B) and the functions (g_A, g_B):

$$\int_0^{2\pi} \hat{U}_A(x)g'_B(x)dx = -\int_0^{2\pi} \frac{g'_A(x)\alpha(x)}{\omega_A^2}g'_B(x)dx \tag{111}$$

$$= -\left[\frac{g'_A(x)\alpha(x)}{\omega_A^2}g_B(x)\right]_0^{2\pi} - \int_0^{2\pi} g_A(x)g_B(x)dx = -\delta_A^B. \tag{112}$$

These equalities result from periodicity in x and from the differential equation (100) for g_A. δ_A^B is the Kronecker delta, equal to unity if $A = B$, 0 otherwise.

3.2 Slow evolution of the amplitude

With the previous system, we can at every time find the shape and the frequency of any acoustic mode. However, the evolution of $A(T)$ is not constrained. Since we expect several physical effects that are not described at this order to be involved (as damping or energy transfer to the mean flow), we have to derive equations for the waves at the next order (see Appendix B). The following amplitude equation can then be obtained (see Appendix C):

$$\frac{2}{A\omega_0^{-1}}\frac{\mathrm{d}(A\omega_0^{-1})}{\mathrm{d}T} = -\frac{i\omega_0}{Pe_s h^2}\iint dxdy g(x)\partial_{yy}\hat{\Theta}_1 \tag{113}$$

$$+ \iint dxdy(\partial_x\bar{u}_1 + \partial_y\bar{v}_1)\left[(1-\gamma)g(x)^2 + \frac{g'(x)^2}{\omega_0^2\bar{\rho}_0}\left(\frac{Pe_s}{Re_s} - \frac{1}{2}\right)\right] \tag{114}$$

$$- \frac{1}{2\omega_0^2}\iint dxdy g'(x)^2\left(\bar{u}_1\partial_x\bar{\rho}_0^{-1} + \bar{v}_1\partial_y\bar{\rho}_0^{-1}\right). \tag{115}$$

The terms in the right-hand side of this equation describe thermal damping, energy exchange with the mean flow, and heat transfer at the boundaries. The quantity on the left-hand side, $A\omega_0^{-1}$, is related to the energy of the wave-field, that is at the leading order

$$\iint dxdy\bar{\rho}_0\frac{A(T)^2}{2} \times \frac{\hat{u}_1^2}{2} = \frac{A(T)^2}{4\omega_0^4}\int dx g'(x)^2\alpha(x) = \left(\frac{A(T)}{2\omega_0}\right)^2. \tag{116}$$

We emphasize that, in order to describe phenomena that fully rely on baroclinic driving, we have disregarded in this problem dissipation in the boundary layers. Although the associated effective slip boundary condition would be of higher order in the reduced system we consider, dissipation in the boundary layer would however dominate all the terms in (113). Indeed, an order of magnitude of the dimensional time-scale τ_{BL} over which the dissipation in the boundary layer damps the wave can be found by

$$\tau_{BL} \sim \frac{\iint dxdy\rho_*(\tilde{u}')^2}{\iint dxdy\rho_*\nu(\tilde{u}'/\delta_{BL})^2} \sim \frac{h\sqrt{R_s}}{\omega_*\sqrt{\epsilon}}. \tag{117}$$

Therefore, this time-scale is in between the fast one and the slow one. If we were to consider boundary layers, the input power would have to strictly balance such dissipation by viscosity at this time-scale, and this power would at order T result in an additional heat input at $y = 0$ and $y = 1$ as a consequence of viscous heating.

4 Linear response

We now consider the simplest regime of this system, in which an acoustic mode of given amplitude and geometry drives a laminar mean flow. For consistency, we hope that this flow will, in return, not much affect the geometry of the acoustic mode.

4.1 Reynolds stress

For this problem, $\bar{\rho}_0 = T_B^{-1} = (1 + \Gamma y)^{-1}$, so that

$$\alpha = 1 + \frac{\Gamma}{2} \tag{118}$$

does not depend on x. We can therefore find an explicit expression for the function g,

$$g(x) = C_1 \cos\left(\frac{\omega_0}{\sqrt{\alpha}}x\right) + C_2 \sin\left(\frac{\omega_0}{\sqrt{\alpha}}x\right), \tag{119}$$

where C_1 and C_2 are constants that have to be determined. The boundary conditions on g, $g'(0) = g'(2\pi) = 0$, fix $C_2 = 0$. Moreover, periodicity requires

$$\omega_0 = n\sqrt{\alpha}, \tag{120}$$

and, with the normalization condition, we end up with

$$g(x) = \frac{\cos(nx)}{\sqrt{\pi}}. \tag{121}$$

We can then compute $\hat{u}_1$ and $\hat{v}_1$. $\hat{u}_1$ is

$$\hat{u}_1 = -\frac{g'}{\omega_0^2\bar{\rho}_0} = (1 + \Gamma y)\frac{\sin(nx)}{n\alpha\sqrt{\pi}}, \tag{122}$$

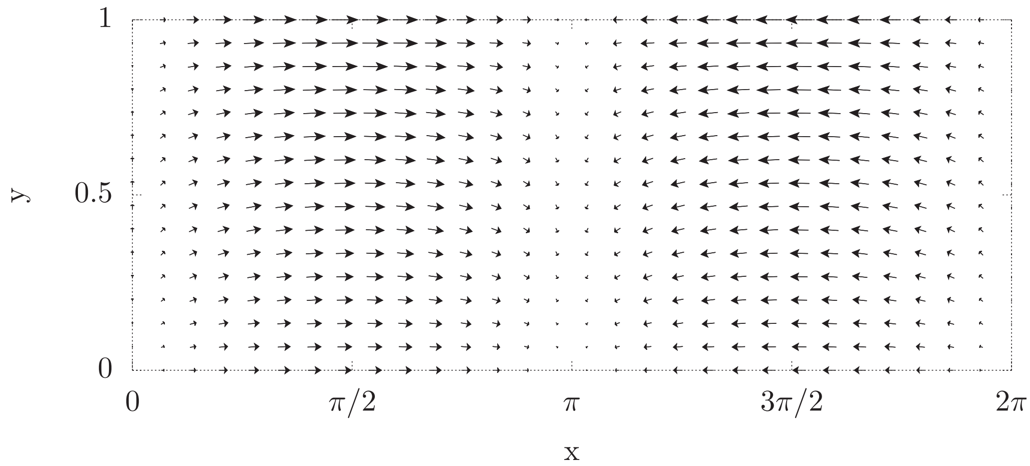

Figure 5: Geometry of the acoustic velocity fields for the mode $n = 1$.

and $\hat{v}_1$ is

$$\hat{v}_1 = y\frac{\cos(nx)}{\sqrt{\pi}} - \partial_x\left(\frac{n\sin(nx)}{n^2\alpha\sqrt{\pi}}\int_0^y(1+\Gamma y)dy\right) = \frac{y}{\sqrt{\pi}}\left(1 - \frac{(1+\Gamma y/2)}{\alpha}\right)\cos(nx). \quad (123)$$

According to (122) and (123), the vertical velocity field does not depend on n, contrary to the horizontal one that decays with n. The velocity field for $n = 1$ is sketched in Fig. 5, and reveals the presence of vorticity. We then evaluate the first contribution to the Reynolds stress,

$$-\partial_x(\bar{\rho}_0\overline{u_1'^2}) = -\partial_x\left(\frac{1}{1+\Gamma y}\frac{A(T)^2}{2}\hat{u}_1^2\right) = -A(T)^2\frac{(1+\Gamma y)}{2n\alpha^2\pi}\sin(2nx), \quad (124)$$

the second one,

$$-\partial_y(\bar{\rho}_0\overline{u_1'v_1'}) = -\partial_y\left(\frac{1}{1+\Gamma y}\frac{A(T)^2}{2}\hat{u}_1\hat{v}_1\right) = \frac{A(T)^2}{4\pi n\alpha^2}\Gamma\left(y - \frac{1}{2}\right)\sin(2nx), \quad (125)$$

so that the full Reynolds stress reads

$$\boxed{R(x,y) = -\frac{A(T)^2}{2\pi n\alpha^2}\left(1 + \frac{\Gamma}{4} + \frac{\Gamma}{2}y\right)\sin(2nx)} \quad (126)$$

4.2 Laminar mean flow

We then assume that the steady state mean flow is small enough so that equations can be linearized. Contrary to the usual analysis of Rayleigh streaming, we cannot use a stream function because, in general, this flow is compressible (even though the Mach number is negligible). The governing equations (63-67) read

$$\begin{cases} 0 = -\frac{1}{\gamma}\partial_x \delta\bar{\pi} + R(x,y) + \frac{\partial_{yy}\delta\bar{u}}{Re_s h^2} & (127) \\ \partial_y \delta\bar{\pi} = 0 & (128) \\ \partial_x \delta\bar{u} + \partial_y \delta\bar{v} = \frac{\partial_{yy}\delta\bar{\Theta}_0}{Pe_s h^2} & (129) \\ \delta\bar{v} = (1+\Gamma y)\frac{\partial_{yy}\delta\bar{\Theta}}{\Gamma Pe_s h^2} & (130) \\ \delta\bar{\rho} = -\frac{\delta\Theta}{(1+\Gamma y)^2} & (131) \end{cases}$$

We can remove $\bar{\pi}$ by combining (127) and (128),

$$\partial_{xy} R = -\frac{\partial_{yyyx}\delta\bar{u}}{Re_s h^2}. \tag{132}$$

The conservation of mass (129) and (130) also imply

$$\partial_x \delta\bar{u} = -\frac{(1+\Gamma y)}{\Gamma Pe_s h^2}\partial_{yyy}\delta\bar{\Theta}. \tag{133}$$

Finally, we obtain a close partial differential equation for $\bar{\Theta}$,

$$\partial_{xy} R = \frac{(1+\Gamma y)\partial_{yyyyyy}\delta\bar{\Theta} + 3\Gamma\partial_{yyyyy}\delta\bar{\Theta}}{\Gamma Re_s Pe_s h^4}, \tag{134}$$

that is,

$$\boxed{\frac{(1+\Gamma y)\partial_{yyyyyy}\delta\bar{\Theta} + 3\Gamma\partial_{yyyyy}\delta\bar{\Theta}}{\Gamma Re_s Pe_s h^4} = -\frac{2A^2\Gamma}{\pi(2+\Gamma)^2}\cos(2nx)} \tag{135}$$

This equation has to be solved with the following boundary conditions

1. $\delta\bar{\Theta}(y=0) = \delta\bar{\Theta}(y=0) = 0$ (fixed temperatures at the boundaries)
2. $\delta\bar{\Theta}''(y=0) = \delta\bar{\Theta}''(y=1) = 0$ (from (130))
3. $\delta\bar{\Theta}'''(y=0) = \delta\bar{\Theta}'''(y=1) = 0$ (from (133))
4. 2π periodicity in x

With these boundary conditions, a unique solution can be found. Since it is quite lengthy, the general solution is postponed to Appendix D and we thereafter focus on $\Gamma = 1$, in which case

$$\delta\bar{\Theta}(x,y) = -\frac{2A^2 Re_s Pe_s h^4}{9\pi}\cos(2nx)G(y), \tag{136}$$

with

$$\begin{aligned} G(y) =& \frac{1}{1080(-3+\ln(16))} \times \left[60(1+y)^2\ln(1+y)\right. \\ &\left. + y\left(94 - 90y - 20y^2 - 222\ln(2) + 3y^4(-3+\ln(16)) - 5y^3(-5+\ln(64))\right)\right]. \end{aligned} \tag{137}$$

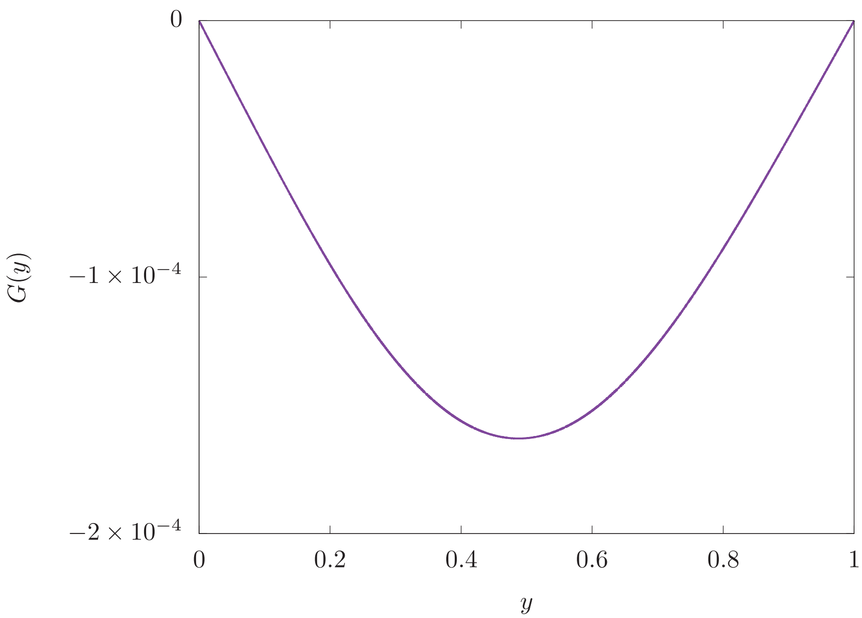

Figure 6: Vertical structure of the temperature perturbation.

This function is always negative, and reaches its minimal value in the range $[0, 1]$ at $y = 0.488$, where $G \simeq -1.63 \times 10^{-4}$. It turns out to be quite similar to a sine function, see Fig. 6. The very small values taken by this function in this range may seem surprising given that it is the solution of an ordinary differential equation with coefficients of order unity. However, similar features often arise when high order derivatives cap the variation speed of a function with boundary condition zero at both sides (see, for instance, the linear regime of convection between two vertical walls of different temperatures).

With this result, we can compare the amplitude of the velocity fields balanced by viscosity caused by dissipation in the boundary layers (Rayleigh streaming) to the one driven by a background density gradient. The dimensional maximal velocities of these flows along the x direction are respectively[5]

$$\tilde{u}_R = \frac{3U_*^2}{8a_*} = \left(\frac{3\epsilon}{8}\right) U_* \qquad \text{and} \qquad \tilde{u}_B = \left(3.2 \times 10^{-4} A^2 Re_s h^2\right) U_*. \tag{138}$$

Therefore, for small values of ϵ, the baroclinic forcing dominates the one caused by dissipation in the boundary layers.

4.3 Comparison with previous work

This streaming flow can be compared to the study of Lin and Farouk [8], in which direct numerical simulations of the full set of equations have been performed. The system considered is similar to ours and consists in a thin channel in which a temperature difference

[5]Numerically, $\max_{y\in[0,1]} |(1 + y)G'''(y)| = 8.99 \times 10^{-3}$ (reached for $y = 0.777$).

can be applied between the two horizontal walls. With no thermal driving, the response is found close to the one described in Sec. 1 and consists of stack cells. When a temperature difference is applied, the vertical cells merge.

For their case 1C, corresponding to the highest temperature difference, the dimensionless parameters are:

$$\epsilon = 10^{-2}, \quad \gamma = 1.4, \quad \Gamma = 0.2, \quad h = 2.3, \quad Re_s = 5.7, \quad Pe_s = 4.1. \tag{139}$$

The amplitude A of the wave field is not reported, but it can be reasonably assumed to be the one in the absence of temperature difference, in which case $A \simeq 6$.

For these parameters, the x (resp. y) component of the streaming velocity at $x = 3\pi/4$ (resp. $\pi/2$) are found to be in our model

$$\delta\bar{u}(x = \frac{3\pi}{4}, y) = -\frac{(1+\Gamma y)}{\pi(2+\Gamma)^2} A^2 Re_s h^2 \Gamma G_\Gamma'''(y), \tag{140}$$

and

$$\delta\bar{v}(x = \frac{\pi}{2}, y) = \frac{(1+\Gamma y)}{\pi(2+\Gamma)^2} 2A^2 Re_s h^2 \Gamma G_\Gamma''(y). \tag{141}$$

where the function G now reads

$$G_{0.2}(y) = 47.664(7.19999866y - 0.1575906y^2 + 0.096y^3 - 0.0047285y^4 \tag{142}$$

$$+ 0.0002914y^5 - 1.44y^2 \ln(0.2) - (36 + 14.4y + 1.44y^2)\ln(1 + 0.2y)) \tag{143}$$

The data of [8] are not dimensionless, and we then consider the dimensional streaming velocities, obtained by multiplying $\delta\bar{u}$ by ϵa_*, and $\delta\bar{v}$ by $\epsilon^{3/2} a_* h$, with $a_* = 353\ \mathrm{m\cdot s^{-1}}$ here. The comparison is reported in Fig. 7, and shows a quantitative agreement (no adjustable parameter), although the dynamics in [8] involves several effects not taken into account by the linear response model, as boundary layers, viscous heating, inertia, and evolution of the viscosity and diffusivity with temperature. This confirms that our model captures the main features of this dynamics.

4.4 Additional heat flux

Fig. 8 shows the streaming velocity field together with the temperature perturbation: we can clearly infer an increase in the heat flux, that we want to compute. A practical issue we face is that the first order additional heat flux vanishes, since the temperature disturbance is of zero mean in the x direction:

$$\kappa \int_0^{2\pi} dx (\partial_y \delta\bar{\Theta})(x, y = 0) \propto \int_0^{2\pi} dx \cos(2nx) = 0. \tag{144}$$

Instead of computing the next order temperature perturbation, we will show that the integral of $(\tilde{\nabla}\tilde{T})^2$ contains the information we are looking for. To prove this, we go back to the dimensionless quantities and the expansion in power of ϵ. On one hand, integration by part of this integral yields

$$\kappa \iint (\tilde{\nabla}\tilde{T})^2 dxdy = \kappa \iint dxdy (\tilde{\nabla}\tilde{T}_B + \tilde{\nabla}\tilde{\Theta}_0 + \epsilon\tilde{\nabla}\tilde{\Theta}_1 + \dots)^2 \tag{145}$$

$$= \kappa \iint dxdy \left((\tilde{\nabla}\tilde{T}_B)^2 + (\tilde{\nabla}\tilde{\Theta}_0)^2 + \epsilon^2 (\tilde{\nabla}\tilde{\Theta}_1)^2 + \dots \right). \tag{146}$$

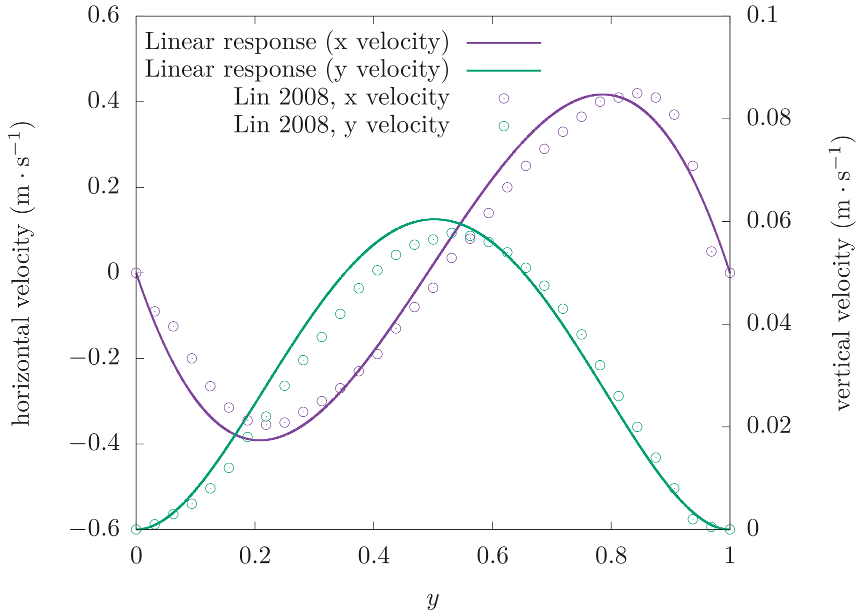

Figure 7: Comparison between the numerical data of [8] (case 1C) and the linear response.

On the other hand, we also have

$$\kappa \iint (\tilde{\nabla}\tilde{T})^2 dxdy = \kappa \int dx \left[\tilde{T}\partial_{\tilde{y}}\tilde{T}\right]_0^{H_*} - \kappa \iint dxdy \tilde{T}\tilde{\nabla}^2\tilde{T} = \dot{Q}\Delta\Theta_* - \kappa \iint dxdy \tilde{T}\tilde{\nabla}^2\tilde{T}, \tag{147}$$

where $\Delta\Theta_* = \tilde{T}(\tilde{y} = H_*) - \tilde{T}(\tilde{y} = 0)$ is the imposed temperature difference and $\dot{Q} > 0$ the heat flux that goes through this system (see (74)). We then work out the last term with the heat equation, with viscous heating denoted as Φ, as in (43).

$$\kappa \iint dxdy \tilde{T}\tilde{\nabla}^2\tilde{T} = \iint dxdy \tilde{T}\left(\tilde{\rho}c_v \frac{D\tilde{T}}{D\tilde{t}} + \tilde{p}(\tilde{\nabla}\cdot\tilde{\mathbf{u}}) + \Phi\right) \tag{148}$$

$$= \frac{c_v}{2}\iint dxdy \tilde{\rho}(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{T}^2 - \iint \tilde{T}\tilde{p}\frac{(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{\rho}}{\tilde{\rho}} dxdy + \iint dxdy \Phi T \tag{149}$$

$$= 0 - \iint \frac{\tilde{p}}{\tilde{\rho}}\left((\tilde{\mathbf{u}}\cdot\tilde{\nabla})(\tilde{T}\tilde{\rho}) - \tilde{\rho}(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{T}\right) dxdy + \iint dxdy \Phi\tilde{T} \tag{150}$$

To derive these equations, we explicitly state that we consider a steady-state, for which $\tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}}) = 0$. Using the equality $\iint dxdy \tilde{\rho}\tilde{T}(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{T} = 0$ derived in this set of equations, we obtain

$$\kappa \iint dxdy \tilde{T}\tilde{\nabla}^2\tilde{T} = -\iint dxdy \tilde{T}\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{p} + \iint dxdy \tilde{p}(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{T} + \iint dxdy \Phi\tilde{T} \tag{151}$$

$$= -\iint dxdy \tilde{T}\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{p} + \iint dxdy \Phi\tilde{T}. \tag{152}$$

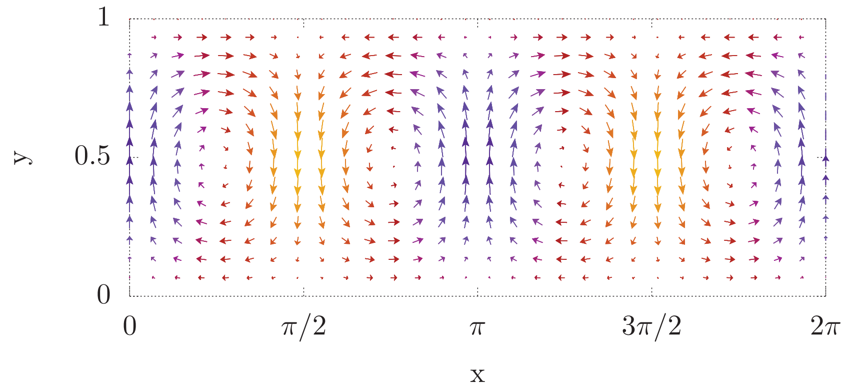

Figure 8: Mean velocity field forced by the mode $n = 1$. Color is the supplementary temperature perturbation (blue is cold): remember that the bottom boundary is cold in this setup.

We recognize the pressure gradient of the momentum equation, so that

$$-\kappa \iint dxdy\tilde{T}\tilde{\nabla}^2\tilde{T} = \iint dxdy\tilde{T}\left(\tilde{\mathbf{u}}\cdot(\tilde{\nabla}\tilde{p}) - \Phi\right) \tag{153}$$

$$= \iint dxdy\tilde{T}\left(\tilde{\mathbf{u}}\cdot\mu\left[\tilde{\nabla}^2\tilde{\mathbf{u}} + \frac{1}{3}\tilde{\nabla}(\tilde{\nabla}\cdot\tilde{\mathbf{u}})\right] - \tilde{\mathbf{u}}\cdot(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{\mathbf{u}} - \Phi\right). \tag{154}$$

If the temperature were constant, viscous heating would cancel out the viscous term, see (46). For the given study, all these terms are of high order in ϵ, and can then be neglected. Therefore, at the leading order, the heat flux in a steady state is given by

$$\dot{Q}\Delta\Theta_* = \kappa \iint (\tilde{\nabla}\tilde{T})^2 dxdy. \tag{155}$$

To deal with dimensionless quantities, we introduce the Nusselt number Nu for the heat flux, defined by

$$Nu = \frac{\dot{Q}}{2\pi\kappa\Delta\Theta_*(k_*H_*)^{-1}} = 1 + \frac{1}{2\pi\Gamma}\int_0^{2\pi} dx(\partial_y(\Theta_0 + \epsilon\Theta_1 + \dots))(y=0). \tag{156}$$

From (155) and (156), we obtain at the leading order in ϵ

$$\boxed{Nu - 1 = \frac{1}{2\pi\Gamma}\int_0^{2\pi} dx(\partial_y\bar{\Theta}_0)(y=0) = \frac{1}{2\pi\Gamma^2}\iint dxdy(\partial_y\bar{\Theta}_0)^2} \tag{157}$$

In this section, $\bar{\Theta}_0$ has been computed for $\Gamma = 1$ as a linear response at the leading order in the wave amplitude $A(T)$, see (136). Thus, [6]

$$Nu - 1 \underset{\Gamma=1}{=} \frac{2A^4 Re_s^2 Pe_s^2 h^8}{81\pi^2}\int_0^1 dy G'_{BC1}(y)^2 \underset{\Gamma=1}{\simeq} 3.2A^4 Re_s^2 Pe_s^2 h^8 10^{-10}. \tag{158}$$

[6]Numerics give $\int_0^1 dyG'_{BC1}(y)^2 \simeq 1.276 \times 10^{-7}$.

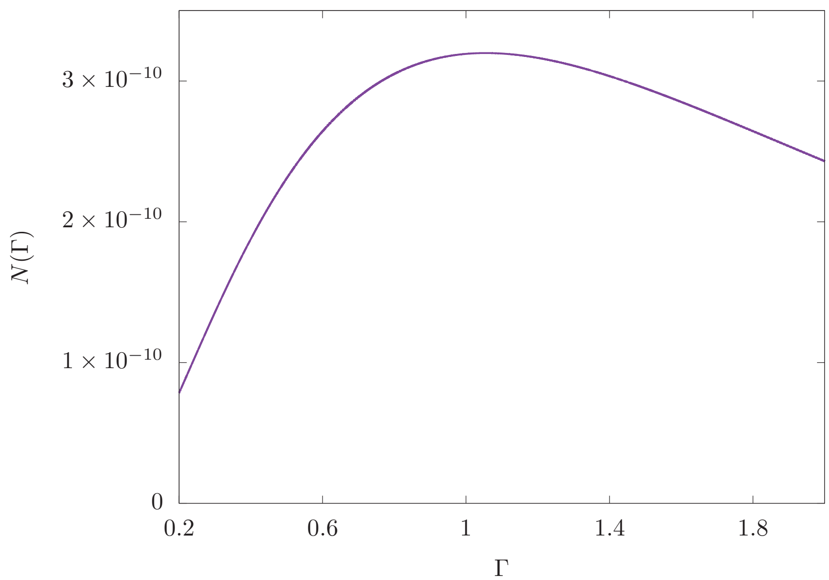

Figure 9: Prefactor of the Nusselt number

Generally speaking, it can be cast under the form

$$Nu - 1 = N(\Gamma)A^4 Re_s^2 Pe_s^2 h^8, \tag{159}$$

where $N(\Gamma)$ is reported in Fig. 9. This has to be compared to the result obtained for streaming based on dissipation in the boundary layers, worked out in [6], that reads[7]

$$(Nu-1)_{BL} = 6.2 Pe_s^2 h^4 \epsilon^2 10^{-6}. \tag{160}$$

This demonstrates that, although the Nusselt number we just derived is very small, it still lies several order of magnitude above the one resulting from acoustic streaming in the boundary layers. Our very small Nusselt number is a consequence of the fact that the Reynolds stress is balanced for this linear solution by viscosity, and not by inertia. This second regime is associated with larger velocities, hence larger heat conduction.

We emphasize, again, that the present study neglects dissipation in the boundary layers. Would it be considered, the additional heat flux would be dominated by the power required to sustain the wave.

4.5 Efficiency of this heat pump

We look for the power required to sustain the linear steady state. This can be done with the amplitude equation. In the limit of small amplitude, the dominant term is the one that does not vanish for $\bar{u}_1 = \bar{v}_1 = \bar{\Theta}_1 = 0$. For $\Gamma = 1$, we find

$$\hat{\Theta}_1 = \frac{i\cos(x)}{\omega_0\sqrt{\pi}}\left((\gamma-1) + y(\gamma - \frac{2}{3}) - \frac{y^2}{3}\right), \tag{161}$$

[7] The Péclet number PE defined in [6] is in our notations $PE = 3Pe_s h^2\epsilon/32$.

and then

$$-\frac{i\omega_0}{Pe_s h^2}\iint dxdy g(x)\partial_{yy}\hat{\Theta}_1 = -\frac{2}{3Pe_s h^2}. \tag{162}$$

This stands for dissipation caused by thermal diffusion. In order to observe a steady-state, this has to be balanced by an input power $\mathcal{P}$, so that the amplitude equation finally reads

$$\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}T} = -\frac{2}{3Pe_s h^2} + \frac{\mathcal{P}}{E}, \tag{163}$$

where $E = A^2/(2\omega_0^2)$ is the mean energy of the acoustic waves. This yields in a steady state

$$P = \frac{2A^2}{9Pe_s h^2}. \tag{164}$$

Finally, the efficiency $\mathcal{E}$ is given by

$$\boxed{\mathcal{E} = \frac{2\pi\Gamma(Nu-1)}{\mathcal{P}} \underset{\Gamma=1}{=} 0.9\times 10^{-8} A^2 Re_s^2 Pe_s^3 h^8} \tag{165}$$

The dependence on A comes from the fact that most of the injected power is, for small A, actually used to balance linear damping of the waves (and not energy transfer from the waves to the mean flow). Although definitely small, this efficiency presents a huge dependence on h that emphasizes the importance on cells of aspect ratio of order one.

5 Numerical Simulations

The linear response solution previously derived, and upon which the additional heat flux and Nusselt number have been computed, does not hold when there is inertia or feedback to the acoustic wave field. In order to describe the dynamics of the system in this regime, we performed numerical simulations of the reduced set of equations (63 - 66), where the Reynolds stress is computed at each time step by solving the eigenvalue problem (100). This has been done with Dedalus [15].

For parameters of order unity, the solution obtained is a steady-state very similar to the linear response. This has been used to check the correctness of our theoretical computa-tions. On the other hand, when these dimensionless parameters are increased, the numerical solution differs from the linear response, and states with strong feedback can be described. The characterization of this regime is still under progress, but as a preliminary result we report the solution obtained for the parameters of Lin *et al.* in Fig. 10.

Conclusion

One main conclusion of this work is that acknowledging the presence of a density gradient as the main driving mechanism of acoustic streaming is crucial to get the correct dynamics in stratified flows. In particular, baroclinic acoustic streaming has been shown to result in velocity fields much bigger than the ones obtained from the usual boundary layer theory. This mechanism is especially favorable for enhancing heat transfers with a good efficiency:

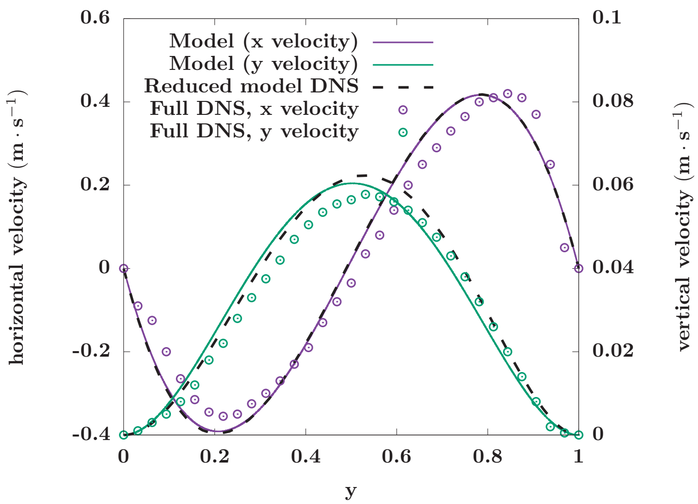

Figure 10: Numerical solution for the parameters of [8] and comparison with the linear model.

whereas the power used to excite the acoustic waves is mainly converted to heat within the boundary layers in the absence of temperature driving, it can *a priori* be fully transferred to the streaming flow with the baroclinic mechanism.

Because streaming motions are important in this regime, the temperature disturbance is also large, which results in a two-way coupling between the waves and the mean flow [10]. For a strongly stratified thin layer, the dynamics involve both a fast time (associated with the period of the waves) and a slow one (on which the mean quantities evolve). With a solvability condition, we have completely eliminated the fast time scale, and found that both the geometry and the amplitude of the waves can be explicitly solved on the slow time scale. We have also obtained a first order solution for the steady-state that compares well with the previous numerical study of the full system with moderate aspect ratio. Numerical simulations can be used to obtain the solution for any range of parameters. This has been done to check that our theoretical computations were correct, and to compare them to the DNS of Lin *et al.* [8].

Moreover, the computation of the heat flux and of the efficiency clearly points out the relevance of aspect ratios of order one. In such limit, the flow associated with acoustic streaming caused by boundary layers is restricted to a small portion of the domain, whereas the one resulting from baroclinic acoustic streaming, acting as a bulk force, would probably not. More generally, such quasi-linear systems in which fast waves are coupled with the mean flow they drive, can be found in other domains of fluid mechanics. For instance, it describes zonal jets (see [16] and references therein) and strongly stratified turbulence [17].

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References

[1] M.S. Longuet-Higgins, Mass transport in water waves, *Phil. Trans. A* **245**, 535 (1953)

[2] J. Lighthill, Acoustic streaming, *J. Sound Vib.* **61**, 391 (1978)

[3] M. Legay *et al.*, Enhancement of Heat Transfer by Ultrasound: Review and Recent Advances, *Int. J. Chem. Eng.* **2011**, 670108 (2011)

[4] L. Rayleigh, On the circulation of air observed in Kundts tubes, and on some allied acoustical problems, *Philos. Trans. R. Soc. London* **175**, 1 (1884).

[5] M. F. Hamilton, Y. A. Ilinskii, and E. A. Zabolotskaya, Acoustic streaming generated by standing waves in two-dimensional channels of arbitrary width, *J. Acoust. Soc. Am.* **113**, 153 (2003)

[6] P. Vainshtein, M. Fishman and C. Gutfinger, Acoustic enhancement of heat transfer between two parallel plates, *Int. J. Heat Mass Transf.* **38**, 1893 (1995)

[7] B-G. Loh, S. Huan, P. I. Ro and C. Kleinstreuer, Acoustic streaming induced by ultrasonic flexural vibrations and associated enhancement of convective heat transfer, *J. Acoust. Soc. Am.* **111**, 875 (2002)

[8] Y. Lin and B. Farouk, Heat transfer in a rectangular chamber with differentially heated horizontal walls: Effects of a vibrating sidewall, *Int. J. Heat Mass Transf.* **51**, 3179 (2008)

[9] M. K. Atkas and T. Ozgumus, The effects of acoustic streaming on thermal convection in an enclosure with differentially heated horizontal walls, *Int. J. Heat Mass Transf.* **53**, 5289 (2010)

[10] G. P. Chini, Z. Malecha and T. D. Dreeben, Large-amplitude acoustic streaming, *J. Fluid Mech.* **744**, 329 (2014)

[11] T. D. Dreeben and G. P. Chini, Two-dimensional streaming flows in high-intensity discharge lamps, *Phys. Fluids* **23**, 056101 (2011)

[12] C. Eckart, Vortices and streams caused by sound waves, *Phys. Rev.* **73**, 68 (1948)

[13] H. Medwin and I. Rudnick, Surface and volume sources of vorticity in acoustic fields, *J. Acoust. Soc. Am.* **25**, 538 (1953)

[14] J. T. Stuart, Double boundary layers in oscillatory viscous flow, *J. Fluid Mech.* **24**, 673 (1966)

[15] K. J. Burns, G. M. Vasil, J. S. Oishi, D. Lecoanet, B. P. Brown, and E. Quataert, Dedalus: A Flexible Framework for Spectrally Solving Differential Equations (in preparation)

[16] J. B. Marston, G. P. Chini and S. M. Tobias, Generalized Quasilinear Approximation: Application to Zonal Jets, *Phys. Rev. Lett.* **116**, 214501 (2016)

[17] G. P. Chini, K. Julien, C. Rocha, C.-C. Caulfield, Coupled reduced equations for strongly stratified flows (in preparation)

A Derivation of the energy equation

The energy balance can be obtained from the initial set of equations (41 - 44), starting from

$$0 = \int_{\mathcal{C}} \mathrm{d}\ell (\tilde{p}\tilde{\mathbf{u}})\mathrm{d}\vec{S} = \iint dxdy \tilde{\nabla}\cdot(\tilde{p}\tilde{\mathbf{u}}), \tag{166}$$

and using the fact that for any function f, our boundary conditions lead to

$$\iint dxdy \frac{Df}{Dt} = \iint dxdy \left(\partial_t f + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})f\right) = \frac{\mathrm{d}}{\mathrm{d}t}\iint dxdy f - \iint dxdy f(\tilde{\nabla}\cdot\tilde{\mathbf{u}}). \tag{167}$$

The main steps of the computation are:

1. Split the divergence in (166) and use (41) and (43) to obtain
$$0 = -\iint \frac{\tilde{p}}{\tilde{\rho}}\frac{D\tilde{\rho}}{D\tilde{t}} - \iint dxdy \tilde{\mathbf{u}}\tilde{\rho}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} + \iint dxdy \Phi. \tag{168}$$

2. With (42), find
$$\iint dxdy \tilde{\mathbf{u}}\tilde{\rho}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} = \frac{d}{d\tilde{t}}\iint \frac{\tilde{\rho}\tilde{\mathbf{u}}^2}{2}. \tag{169}$$

3. Using (42), (43) and $c_v(\gamma-1) = R_s$ for an ideal gas, derive
$$-\iint \frac{\tilde{p}}{\tilde{\rho}}\frac{D\tilde{\rho}}{D\tilde{t}} = -\frac{d}{d\tilde{t}}\iint dxdy \frac{\tilde{p}}{\gamma-1} + \dot{Q} + \iint dxdy \Phi, \tag{170}$$
and finally obtain (73).

B Set of equations for the waves at the next order

NS in the x direction. We look for (68) at the next order. Eq. (48) at order ϵ^2 is

$$\rho_1\left[\omega_0\partial_\phi u_1'\right] + \bar{\rho}_0\left[\omega_0\partial_\phi u_2' + \omega_1\partial_\phi u_1' + \partial_T u_1 + u_1\partial_x u_1 + v_1\partial_y u_1\right] = -\frac{1}{\gamma}\partial_x\pi_2 + \frac{1}{Re_s h^2}\partial_{yy}u_1. \tag{171}$$

Fast-time averaging this equation yields

$$\omega_0\overline{\rho_1'\partial_\phi u_1'} + \bar{\rho}_0\left[\partial_T\bar{u}_1 + \overline{u_1\partial_x u_1} + \overline{v_1\partial_y u_1}\right] = -\frac{1}{\gamma}\partial_x\bar{\pi}_2 + \frac{1}{Re_s h^2}\partial_{yy}\bar{u}_1, \tag{172}$$

that eventually becomes (58) with the use of (70). To get an equation for u_2', we subtract equation (172) to (171),

$$\omega_0\bar{\rho}_0\partial_\phi u_2' + \frac{1}{\gamma}\partial_x\pi_2' = -\omega_0\left[\rho_1\partial_\phi u_1' - \overline{\rho_1'\partial_\phi u_1'}\right] - \omega_1\bar{\rho}_0\partial_\phi u_1'$$
$$- \bar{\rho}_0\left[\partial_T u_1' + u_1\partial_x u_1 - \overline{u_1\partial_x u_1} + v_1\partial_y u_1 - \overline{v_1\partial_y u_1}\right] + \frac{1}{Re_s h^2}\partial_{yy}u_1'. \quad (173)$$

NS in the y direction. We look for (69) at the next order. Eq. (49) at order ϵ is

$$\frac{\partial_y\pi_2}{\gamma h^2} = -\bar{\rho}_0\omega_0\partial_\phi v_1 \Longrightarrow \partial_y\pi_2' = -\gamma h^2\bar{\rho}_0\omega_0\partial_\phi v_1'. \quad (174)$$

Conservation of mass. We look for (70) at the next order. Eq. (51) at order ϵ^2 is

$$(\omega_0\partial_\phi\rho_2 + \omega_1\partial_\phi\rho_1 + \partial_T\rho_1) + \partial_x\left(\bar{\rho}_0 u_2 + \rho_1 u_1\right) + \partial_y\left(\bar{\rho}_0 v_2 + \rho_1 v_1\right) = 0, \quad (175)$$

Fast time average yields

$$\partial_T\bar{\rho}_1 + \partial_x\left(\bar{\rho}_0\bar{u}_2 + \overline{\rho_1 u_1}\right) + \partial_y\left(\bar{\rho}_0\bar{v}_2 + \overline{\rho_1 v_1}\right) = 0, \quad (176)$$

And then,

$$\omega_0\partial_\phi\rho_2' + \partial_x(\bar{\rho}_0 u_2') + \partial_y(\bar{\rho}_0 v_2') = -\omega_1\partial_\phi\rho_1' - \partial_T\rho_1' - \partial_x\left(\rho_1 u_1 - \overline{\rho_1 u_1}\right) - \partial_y\left(\rho_1 v_1 - \overline{\rho_1 v_1}\right). \quad (177)$$

Internal energy balance We look for (71) at the next order. Eq. (50) at order ϵ^2 is

$$\omega_0\partial_\phi\Theta_2 + \omega_1\partial_\phi\Theta_1 + \partial_T\Theta_1 + u_2\partial_x\bar{\Theta}_0 + u_1\partial_x\Theta_1 + v_2\partial_y\bar{\Theta}_0 + v_1\partial_y\Theta_1 + v_2\frac{dT_B}{dy} = (1-\gamma)$$
$$\left[(T_B + \bar{\Theta}_0)(\partial_x u_2 + \partial_y v_2) + \Theta_1(\partial_x u_1 + \partial_y v_1)\right] + \frac{\gamma}{\bar{\rho}_0 Pe_s}\left(\partial_{xx}\bar{\Theta}_0 + \frac{\partial_{yy}\Theta_1}{h^2} - \frac{\rho_1\partial_{yy}\bar{\Theta}_0}{\bar{\rho}_0 h^2}\right),$$

Fast time average yields

$$\partial_T\bar{\Theta}_1 + \bar{u}_2\partial_x\bar{\Theta}_0 + \overline{u_1\partial_x\Theta_1} + \bar{v}_2\partial_y\bar{\Theta}_0 + \overline{v_1\partial_y\Theta_1} + \bar{v}_2\frac{dT_B}{dy} = (1-\gamma) \quad (178)$$
$$\left[(T_B + \bar{\Theta}_0)(\partial_x\bar{u}_2 + \partial_y\bar{v}_2) + \overline{\Theta_1(\partial_x u_1 + \partial_y v_1)}\right] + \frac{\gamma}{\bar{\rho}_0 Pe_s}\left(\partial_{xx}\bar{\Theta}_0 + \frac{\partial_{yy}\bar{\Theta}_1}{h^2} - \frac{\bar{\rho}_1\partial_{yy}\bar{\Theta}_0}{\bar{\rho}_0 h^2}\right),$$

And then

$$\omega_0\partial_\phi\Theta_2' + \omega_1\partial_\phi\Theta_1' + \partial_T\Theta_1' + u_2'\partial_x\bar{\Theta}_0 + u_1\partial_x\Theta_1 - \overline{u_1\partial_x\Theta_1} + v_2'\partial_y\bar{\Theta}_0 + v_1\partial_y\Theta_1 - \overline{v_1\partial_y\Theta_1}$$
$$+ v_2'\frac{dT_B}{dy} = (1-\gamma)\left[(T_B + \bar{\Theta}_0)(\partial_x u_2' + \partial_y v_2') + \Theta_1(\partial_x u_1 + \partial_y v_1) - \overline{\Theta_1(\partial_x u_1 + \partial_y v_1)}\right]$$
$$+ \frac{\gamma}{Pe_s}\left(\frac{\partial_{yy}\Theta_1'}{\bar{\rho}_0 h^2} - \frac{\rho_1'\partial_{yy}\bar{\Theta}_0}{\bar{\rho}_0^2 h^2}\right),$$

that also reads

$$\omega_0\partial_\phi\Theta_2' + u_2'\partial_x\bar{\Theta}_0 + v_2'\partial_y\left(\bar{\Theta}_0 + T_B\right) + (\gamma-1)\left[(T_B + \bar{\Theta}_0)(\partial_x u_2' + \partial_y v_2')\right]$$
$$= -\omega_1\partial_\phi\Theta_1' - \partial_T\Theta_1' - u_1\partial_x\Theta_1 + \overline{u_1\partial_x\Theta_1} - v_1\partial_y\Theta_1 + \overline{v_1\partial_y\Theta_1}$$
$$+ (1-\gamma)\left[\Theta_1(\partial_x u_1 + \partial_y v_1) - \overline{\Theta_1(\partial_x u_1 + \partial_y v_1)}\right] + \frac{\gamma}{\bar{\rho}_0 Pe_s h^2}\left(\partial_{yy}\Theta_1' - \frac{\rho_1'\partial_{yy}\bar{\Theta}_0}{\bar{\rho}_0}\right). \quad (179)$$

Equation of state. We look for (72) at the next order. Eq. (52) at order ϵ^2 is

$$\pi_2 = \rho_2(T_B + \bar{\Theta}_0) + \rho_1\Theta_1 + \bar{\rho}_0\Theta_2, \tag{180}$$

Fast time average yields

$$\bar{\pi}_2 = \bar{\rho}_2(T_B + \bar{\Theta}_0) + \overline{\rho_1\Theta_1} + \bar{\rho}_0\bar{\Theta}_2, \tag{181}$$

And then

$$\pi_2' - \rho_2'(T_B + \bar{\Theta}_0) - \bar{\rho}_0\Theta_2' = \rho_1\Theta_1 - \overline{\rho_1\Theta_1}. \tag{182}$$

C Slow evolution of the amplitude

We wish to derive an equation for the evolution of $A(T)$. To do so, we have to find a solvability condition. We proceed as follows:

1. We write all the variables at the next order as

$$f(x,y,T,\phi)_2' = B(T)/2\left(e^{i\phi}\hat{f}_2(x,y,T) + c.c.\right). \tag{183}$$

   We keep the previous form for the variables at order one, and we discard all the terms that do not go as $e^{\pm i\phi}$.

2. Using the set of equations at the next order, we reduce this system of five variables ($\hat{u}_2$, $\hat{v}_2$, $\hat{\pi}_2$, $\hat{\rho}_2$, $\hat{\Theta}_2$) to a system of two variables ($\hat{u}_2$ and $\hat{v}_2$).

3. We find the adjoint of this linear system and the general form of the solvability condition.

4. We enforce it and get a close equation for $dA(T)/dT$.

C.1 Step 2, part 1: get an expression for $\hat{\rho}_2$

We recall the conservation of mass at the next order, equation (177):

$$\omega_0\partial_\phi\rho_2' + \partial_x(\bar{\rho}_0 u_2') + \partial_y(\bar{\rho}_0 v_2') = -\omega_1\partial_\phi\rho_1' - \partial_T\rho_1' - \partial_x\left(\rho_1 u_1 - \overline{\rho_1 u_1}\right) - \partial_y\left(\rho_1 v_1 - \overline{\rho_1 v_1}\right).$$

It becomes

$$\begin{aligned} &B(T)\left(\omega_0 i\hat{\rho}_2 + \partial_x(\bar{\rho}_0\hat{u}_2) + \partial_y(\bar{\rho}_0\hat{v}_2)\right) = \\ &-\frac{dA}{dT}\hat{\rho}_1 - A(T)\left[\omega_1 i\hat{\rho}_1 + \frac{\delta\hat{\rho}_1}{\delta T} + \partial_x(\bar{\rho}_1\hat{u}_1 + \hat{\rho}_1\bar{u}_1) + \partial_y(\bar{\rho}_1\hat{v}_1 + \hat{\rho}_1\bar{v}_1)\right], \end{aligned} \tag{184}$$

where $\delta\hat{f}/\delta T$ is the functional derivative of $\hat{f}$ with respect to any dependence it may have on slow time T. For a later use, we rewrite this equation as

$$\begin{aligned} \frac{B(T)\hat{\rho}_2}{\bar{\rho}_0} &= \frac{iB(T)}{\omega_0\bar{\rho}_0}\left(\partial_x(\bar{\rho}_0\hat{u}_2) + \partial_y(\bar{\rho}_0\hat{v}_2)\right) + \frac{i}{\omega_0\bar{\rho}_0}\frac{dA}{dT}\hat{\rho}_1 \\ &\quad + \frac{i}{\omega_0\bar{\rho}_0}A(T)\left[\frac{\delta\hat{\rho}_1}{\delta T} + i\omega_1\hat{\rho}_1 + \partial_x(\bar{\rho}_1\hat{u}_1 + \hat{\rho}_1\bar{u}_1) + \partial_y(\bar{\rho}_1\hat{v}_1 + \hat{\rho}_1\bar{v}_1)\right]. \end{aligned} \tag{185}$$

C.2 Step 2, part 2: get an expression for $\hat{\Theta}_2$

We perform the same simplification for the internal energy balance at order two (179),

$$\frac{B(T)}{A(T)}\left(i\omega_0\hat{\Theta}_2 + \hat{u}_2\partial_x\bar{\Theta}_0 + \hat{v}_2\partial_y\left(\bar{\Theta}_0 + T_B\right) + (\gamma-1)\left[(T_B+\bar{\Theta}_0)(\partial_x\hat{u}_2 + \partial_y\hat{v}_2)\right]\right)$$
$$= -\omega_1 i\hat{\Theta}_1 - \frac{1}{A(T)}\frac{dA}{dT}\hat{\Theta}_1 - \frac{\delta\Theta}{\delta T} - \hat{u}_1\partial_x\bar{\Theta}_1 - \bar{u}_1\partial_x\hat{\Theta}_1 - \hat{v}_1\partial_y\bar{\Theta}_1 - \bar{v}_1\partial_y\hat{\Theta}_1$$
$$+ (1-\gamma)\left[\hat{\Theta}_1(\partial_x\bar{u}_1 + \partial_y\bar{v}_1) + \bar{\Theta}_1(\partial_x\hat{u}_1 + \partial_y\hat{v}_1)\right] + \frac{\gamma}{\bar{\rho}_0 Pe_s h^2}\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right), \quad (186)$$

that can also be written

$$B(T)\hat{\Theta}_2 = \frac{B(T)i}{\omega_0}\left(\hat{u}_2\partial_x\bar{\Theta}_0 + \hat{v}_2\partial_y\left(\bar{\Theta}_0 + T_B\right) + (\gamma-1)\left[(T_B+\bar{\Theta}_0)(\partial_x\hat{u}_2 + \partial_y\hat{v}_2)\right]\right)$$
$$+ \frac{iA(T)}{\omega_0}\left(\omega_1 i\hat{\Theta}_1 + \frac{1}{A(T)}\frac{dA}{dT}\hat{\Theta}_1 + \frac{\delta\Theta}{\delta T} + \hat{u}_1\partial_x\bar{\Theta}_1 + \bar{u}_1\partial_x\hat{\Theta}_1 + \hat{v}_1\partial_y\bar{\Theta}_1 + \bar{v}_1\partial_y\hat{\Theta}_1\right)$$
$$+ \frac{iA(T)}{\omega_0}(\gamma-1)\left[\hat{\Theta}_1(\partial_x\bar{u}_1 + \partial_y\bar{v}_1) + \bar{\Theta}_1(\partial_x\hat{u}_1 + \partial_y\hat{v}_1)\right]$$
$$- \frac{iA(T)}{\omega_0}\frac{\gamma}{\bar{\rho}_0 Pe_s h^2}\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right). \quad (187)$$

C.3 Step 2, part 3: get an expression for $\hat{\pi}_2$

We recall the equation of state (182),

$$\pi_2' - \rho_2'(T_B + \bar{\Theta}_0) - \bar{\rho}_0\Theta_2' = \rho_1\Theta_1 - \overline{\rho_1\Theta_1}. \quad (188)$$

Now that we know $\hat{\rho}_2$ and $\hat{\Theta}_2$, we can find $\hat{\pi}_2$. First, with our notations and $(T_B+\bar{\Theta}_0) = \bar{\rho}_0^{-1}$,

$$B(T)\hat{\pi}_2 = B(T)\frac{\hat{\rho}_2}{\bar{\rho}_0} + B(T)\bar{\rho}_0\hat{\Theta}_2 + A(T)\hat{\rho}_1\bar{\Theta}_1 + A(T)\bar{\rho}_1\hat{\Theta}_1. \quad (189)$$

If $A(T) = 0$, this equation would be the same as at order one and read

$$\hat{\pi}_2 = \frac{i\gamma}{\omega_0}(\partial_x\hat{u}_2 + \partial_y\hat{v}_2). \quad (190)$$

We have to complete this equation with forcing terms. We get:

$$B(T)\left(\hat{\pi}_2 - \frac{i\gamma}{\omega_0}(\partial_x\hat{u}_2 + \partial_y\hat{v}_2)\right) = \quad (191)$$
$$\frac{i}{\omega_0\bar{\rho}_0}\frac{dA}{dT}\hat{\rho}_1 + \frac{i}{\omega_0\bar{\rho}_0}A(T)\left[\frac{\delta\hat{\rho}_1}{\delta T} + i\hat{\rho}_1\omega_1 + \partial_x(\bar{\rho}_1\hat{u}_1 + \hat{\rho}_1\bar{u}_1) + \partial_y(\bar{\rho}_1\hat{v}_1 + \hat{\rho}_1\bar{v}_1)\right]$$
$$+ \frac{iA(T)\bar{\rho}_0}{\omega_0}\left(\omega_1 i\hat{\Theta}_1 + \frac{1}{A(T)}\frac{dA}{dT}\hat{\Theta}_1 + \frac{\delta\hat{\Theta}_1}{\delta T} + \hat{u}_1\partial_x\bar{\Theta}_1 + \bar{u}_1\partial_x\hat{\Theta}_1 + \hat{v}_1\partial_y\bar{\Theta}_1 + \bar{v}_1\partial_y\hat{\Theta}_1\right)$$
$$+ \frac{iA(T)\bar{\rho}_0}{\omega_0}(\gamma-1)\left[\hat{\Theta}_1(\partial_x\bar{u}_1 + \partial_y\bar{v}_1) + \bar{\Theta}_1(\partial_x\hat{u}_1 + \partial_y\hat{v}_1)\right]$$
$$- \frac{iA(T)}{\omega_0}\frac{\gamma}{Pe_s h^2}\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right) + A(T)\hat{\rho}_1\bar{\Theta}_1 + A(T)\bar{\rho}_1\hat{\Theta}_1.$$

We write this equation as

$$\hat{\pi}_2 = \frac{i\gamma}{\omega_0}(\partial_x \hat{u}_2 + \partial_y \hat{v}_2) + \frac{\mathcal{H}}{B(T)}, \tag{192}$$

where $\mathcal{H}$ is a known complex-valued function.

C.4 Step 2, part 4: get a system of equations for $\hat{u}_2$ and $\hat{v}_2$

We start with the Navier-Stokes equation on the y direction at the second order (174),

$$B(T)\partial_y \hat{\pi}_2 = -i\gamma h^2 \bar{\rho}_0 \omega_0 A(T)\hat{v}_1 \Longrightarrow B(T)\partial_y \left(\partial_x \hat{u}_2 + \partial_y \hat{v}_2\right) = \frac{i\partial_y \mathcal{H}\omega_0}{\gamma} - h^2 \bar{\rho}_0 \omega_0^2 A(T)\hat{v}_1.$$

Similarly, the Navier-Stokes equation on the x direction at the second order (173) is

$$\begin{aligned}
&\omega_0 \bar{\rho}_0 i B(T)\hat{u}_2 + \frac{iB(T)}{\omega_0}\partial_x \left(\partial_x \hat{u}_2 + \partial_y \hat{v}_2\right) + \frac{\partial_x \mathcal{H}}{\gamma} = \\
&- i\omega_0 \bar{\rho}_1 A(T)\hat{u}_1 - i\omega_1 \bar{\rho}_0 A(T)\hat{u}_1 + \frac{A(T)}{Re_s h^2}\partial_{yy}\hat{u}_1 \\
&- \bar{\rho}_0 \left(\frac{dA}{dT}\hat{u}_1 + A(T)\frac{\delta \hat{u}_1}{\delta T} + A(T)\hat{u}_1 \partial_x \bar{u}_1 + A(T)\bar{u}_1 \partial_x \hat{u}_1 + A(T)\hat{v}_1 \partial_y \bar{u}_1 + A(T)\bar{v}_1 \partial_y \hat{u}_1\right),
\end{aligned} \tag{193}$$

that is,

$$\begin{aligned}
&B(T)\left(\partial_x(\partial_x \hat{u}_2 + \partial_y \hat{v}_2) + \omega_0^2 \bar{\rho}_0 \hat{u}_2\right) = \frac{i\omega_0 \partial_x \mathcal{H}}{\gamma} \\
&- \omega_0^2 \bar{\rho}_1 A(T)\hat{u}_1 - \omega_0 \omega_1 \bar{\rho}_0 A(T)\hat{u}_1 - i\omega_0 \frac{A(T)}{Re_s h^2}\partial_{yy}\hat{u}_1 \\
&+ i\omega_0 \bar{\rho}_0 \left(\frac{dA}{dT}\hat{u}_1 + A(T)\frac{\delta \hat{u}_1}{\delta T} + A(T)\hat{u}_1 \partial_x \bar{u}_1 + A(T)\bar{u}_1 \partial_x \hat{u}_1 + A(T)\hat{v}_1 \partial_y \bar{u}_1 + A(T)\bar{v}_1 \partial_y \hat{u}_1\right).
\end{aligned} \tag{194}$$

C.5 Step 3: general solvability condition

The previous system is of the form

$$\begin{cases}
\partial_x(\partial_x \hat{u}_2 + \partial_y \hat{v}_2) + \omega_0^2 \bar{\rho}_0 \hat{u}_2 = \mathcal{F} & (195) \\
\partial_y(\partial_x \hat{u}_2 + \partial_y \hat{v}_2) = \mathcal{G} & (196)
\end{cases}$$

where $\mathcal{F}$ and $\mathcal{G}$ are complex-valued functions. In this vector space $(\mathbb{R}^2 \to \mathbb{C})^2$, a vector writes

$$\mathbf{V} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}. \tag{197}$$

The linear operator we consider is

$$\mathcal{L} = \begin{pmatrix} \partial_{xx} + \omega_0^2 \bar{\rho}_0 & \partial_{xy} \\ \partial_{xy} & \partial_{yy} \end{pmatrix}, \tag{198}$$

so that the above system is simply $\mathcal{L}\mathbf{V} = \mathbf{F}$. We define a scalar product $(\cdot|\cdot)$ as

$$(\mathbf{V}_A|\mathbf{V}_B) = \int_0^{2\pi} dx \int_0^1 dy \left(\mathbf{V}_A^{\mathrm{T}} \cdot \mathbf{V}_B^*\right) = \int_0^{2\pi} dx \int_0^1 dy \left(\hat{u}_A \hat{u}_B^* + \hat{v}_A \hat{v}_B^*\right), \tag{199}$$

where * stands for the conjugate. We can easily check that $\mathcal{L}$ is self-adjoint ($\mathcal{L} = \mathcal{L}^\dagger$), given the 2π periodicity in x and the kinematic boundary conditions in y ($v(x, y = 0) = v(x, y = 1) = 0$):

$$(\mathcal{L}\mathbf{V}_A|\mathbf{V}_B) = \iint dxdy \left[\left(\partial_{xx}\hat{u}_A + \omega_0^2\bar{\rho}_0\hat{u}_A + \partial_{xy}\hat{v}_A\right)\hat{u}_B^* + \left(\partial_{xy}\hat{u}_A + \partial_{yy}\hat{v}_A\right)\hat{v}_B^*\right] \tag{200}$$

$$= \iint dxdy \left[\left(\partial_{xx}\hat{u}_B^* + \omega_0^2\bar{\rho}_0\hat{u}_B^* + \partial_{xy}\hat{v}_B^*\right)\hat{u}_A + \left(\partial_{xy}\hat{u}_B^* + \partial_{yy}\hat{v}_B^*\right)\hat{v}_A\right] \tag{201}$$

$$= (\mathbf{V}_A|\mathcal{L}\mathbf{V}_B). \tag{202}$$

We also know what vectors are in the kernel of $\mathcal{L}$: it consists of the first order acoustic modes already described, one of them being $\mathbf{V}_1 = (\hat{u}_1, \hat{v}_1)$. Therefore, we must have

$$(\mathbf{F}|\mathbf{V}_1) = 0, \tag{203}$$

that also reads,

$$\boxed{\iint \left(\mathcal{F}(x,y)\hat{u}_1^*(x,y) + \mathcal{G}(x,y)\hat{v}_1^*(x,y)\right) dxdy = 0} \tag{204}$$

C.6 Step 4, part 1: first approach of the solvability condition

Given that $\hat{u}_1$ and $\hat{v}_1$ are real-valued fields, the solvability condition (204) can be easily decomposed in a real and imaginary one. The crucial one concerns the imaginary part of $\mathcal{F} = \mathcal{F}_r + i\mathcal{F}_i$ and $\mathcal{G} = \mathcal{G}_r + i\mathcal{G}_i$, that are respectively

$$\mathcal{F}_i = \frac{\omega_0}{\gamma}\partial_x\mathcal{H}_r - \frac{\omega_0 A(T)}{Re_s h^2}\partial_{yy}\hat{u}_1 \tag{205}$$
$$+ \omega_0\bar{\rho}_0\left(\frac{dA}{dT}\hat{u}_1 + A(T)\frac{\delta\hat{u}_1}{\delta T} + A(T)\hat{u}_1\partial_x\bar{u}_1 + A(T)\bar{u}_1\partial_x\hat{u}_1 + A(T)\hat{v}_1\partial_y\bar{u}_1 + A(T)\bar{v}_1\partial_y\hat{u}_1\right),$$

and,

$$\mathcal{G}_i = \frac{\partial_y\mathcal{H}_r\omega_0}{\gamma}. \tag{206}$$

The terms involving $\mathcal{H}_r$ in the solvability conditions are

$$\frac{\omega_0}{\gamma}\iint dxdy\left(\hat{u}_1\partial_x\mathcal{H}_r + \hat{v}_1\partial_y\mathcal{H}_r\right) = -\frac{\omega_0}{\gamma}\iint dxdy\mathcal{H}_r(\partial_x\hat{u}_1 + \partial_y\hat{v}_1) = -\frac{\omega_0}{\gamma}\iint dxdyg(x)\mathcal{H}_r, \tag{207}$$

where the real part of $\mathcal{H}$ results from (191). For this purpose, remember that both $\hat{\rho}_1$, $\hat{\Theta}_1$ and $\hat{\pi}_1$ are pure imaginary fields. Thus,

$$\begin{aligned}\mathcal{H}_r &= \frac{i}{\omega_0\bar{\rho}_0}\frac{dA}{dT}\hat{\rho}_1 + \frac{i}{\omega_0\bar{\rho}_0}A(T)\left[\frac{\delta\hat{\rho}_1}{\delta T} + \partial_x(\hat{\rho}_1\bar{u}_1) + \partial_y(\hat{\rho}_1\bar{v}_1)\right] \\ &+ \frac{iA(T)\bar{\rho}_0}{\omega_0}\left(\frac{1}{A(T)}\frac{dA}{dT}\hat{\Theta}_1 + \frac{\delta\hat{\Theta}_1}{\delta T} + \bar{u}_1\partial_x\hat{\Theta}_1 + \bar{v}_1\partial_y\hat{\Theta}_1\right) \\ &+ \frac{iA(T)\bar{\rho}_0}{\omega_0}(\gamma-1)\left[\hat{\Theta}_1(\partial_x\bar{u}_1 + \partial_y\bar{v}_1)\right] - \frac{iA(T)}{\omega_0}\frac{\gamma}{Pe_s h^2}\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right).\end{aligned} \tag{208}$$

Note that this procedure has removed all the higher order mean-flow terms ($\bar{\Theta}_1$, $\bar{\rho}_1$). This has also removed ω_1, and if we keep to this part of the solvability condition we will not be able to get the slow evolution of the phase. Now, we are going to simplify $\mathcal{H}_r$ with the expressions of $\hat{\rho}_1$ and $\hat{\Theta}_1$. For instance, with (91) and (92),

$$\frac{i}{\omega_0\bar{\rho}_0}\frac{dA}{dT}\hat{\rho}_1 + \frac{i\bar{\rho}_0}{\omega_0}\frac{dA}{dT}\hat{\Theta}_1 = \frac{i}{\omega_0}\frac{dA}{dT}\left(\frac{\hat{\rho}_1}{\bar{\rho}_0} + \bar{\rho}_0\hat{\Theta}_1\right) = \frac{i}{\omega_0}\frac{dA}{dT}\hat{\pi}_1 = -\frac{\gamma}{\omega_0^2}\frac{dA}{dT}g(x). \tag{209}$$

C.7 Step 4, part 2: a closer look at the functional derivatives

For further simplifications, we have to give a formal definition of the functional derivative $\delta/\delta T$. For a given real-valued functional F defined by

$$F\begin{cases}\text{function space} \to \mathbb{R} & (210)\\ f \to F(f) & (211)\end{cases}$$

the functional derivative $\dfrac{\delta F}{\delta f}(h)$ describes how $F(f)$ evolves when $f \to f+h$, with $|h| \to 0$:

$$\frac{\delta F}{\delta f}(h) = \lim_{\varepsilon\to 0}\left(\frac{F(f+\varepsilon h) - F(f)}{\varepsilon}\right). \tag{212}$$

In the present work, we consider functions f that have a functional dependence on $\bar{\rho}_0$ and also depend on the real parameters x and y. In all the previous calculations, we were interested in how the real number $f(x,y;[\bar{\rho}_0])$ evolves for fixed x and y, while $\bar{\rho}_0$ evolves on the slow time. Formally, we should therefore define a functional $f_{x,y}$, such that

$$f_{x,y}\begin{cases}([0,2\pi]\times[0,1] \to \mathbb{R}) \to \mathbb{R} & (213)\\ \bar{\rho}_0 \to f(x,y;[\bar{\rho}_0]) & (214)\end{cases}$$

and the shorthand $\dfrac{\delta f}{\delta T}$ should therefore be understood as

$$\left(\frac{\delta f}{\delta T}\right) \to \frac{\delta f_{x,y}}{\delta\bar{\rho}_0}(\partial_T\bar{\rho}_0). \tag{215}$$

This term is the increment in $f(x,y;[\bar{\rho}_0])$ at fixed x and y for an infinitely small slow-time increase ($\bar{\rho}_0 \to \bar{\rho}_0 + \varepsilon\partial_T\bar{\rho}_0$). We can perform basic operations on these functional derivatives. Let us for instance take the example of one showing up in (208):

$$\left(\frac{\delta\hat{\rho}_1}{\delta T}\right) \to \frac{\delta\hat{\rho}_{1,x,y}}{\delta\bar{\rho}_0}(\partial_T\bar{\rho}_0), \tag{216}$$

with similar notations as above. Moreover, equations (91), (92) and (96) give

$$\hat{\rho}_1 = \frac{i\gamma\bar{\rho}_0 g}{\omega_0} - \bar{\rho}_0^2\hat{\Theta}_1. \tag{217}$$

In this equation, both $\bar{\rho}_0$, g, $\hat{\Theta}_1$ and ω_0 have a functional dependence on $\bar{\rho}_0$. Functional derivatives obey linearity and product rule, so that (216) becomes

$$\left(\frac{\delta\hat{\rho}_1}{\delta T}\right) = i\gamma\left(\frac{\bar{\rho}_0}{\omega_0}\frac{\delta g}{\delta T} + \frac{g}{\omega_0}\frac{\delta\bar{\rho}_0}{\delta T} + \bar{\rho}_0 g\frac{\delta\omega_0^{-1}}{\delta T}\right) - \bar{\rho}_0^2\frac{\delta\hat{\Theta}_1}{\delta T} - \hat{\Theta}_1\frac{\delta\bar{\rho}_0^2}{\delta T}. \tag{218}$$

We then detail each of these terms successively.

$\dfrac{\delta g}{\delta T}$: since g does only depend on α, this term is formally defined as

$$\frac{\delta g}{\delta T} = \frac{\delta g_x}{\delta\alpha}(\partial_T\alpha). \tag{219}$$

$\dfrac{\delta\bar{\rho}_0}{\delta T}$: it stands for the evolution of $\bar{\rho}_0$ at given x and y as time goes by, *i.e.*

$$\frac{\delta\bar{\rho}_0}{\delta T} = \frac{\delta\bar{\rho}_{0,x,y}}{\delta\bar{\rho}_0}(\partial_T\bar{\rho}_0) = \partial_T\bar{\rho}_0(x,y,T). \tag{220}$$

$\dfrac{\delta\omega_0^{-1}}{\delta T}$: ω_0 is also a functional of $\bar{\rho}_0$ (or rather α), because it is defined as an eigenvalue of an ode involving α. This term cannot be changed much,

$$\frac{\delta\omega_0^{-1}}{\delta T} = -\frac{1}{\omega_0^2}\frac{\delta\omega_0}{\delta\alpha}(\partial_T\alpha) = -\frac{\partial_T\omega_0}{\omega_0^2}. \tag{221}$$

$\dfrac{\delta\hat{\Theta}_1}{\delta T}$: this term fortunately cancels out with another one in the expression of $\mathcal{H}_r$.

$\dfrac{\delta\bar{\rho}_0^2}{\delta T}$: similarly to (220),

$$\frac{\delta\bar{\rho}_0^2}{\delta T} = 2\bar{\rho}_0\partial_T\bar{\rho}_0(x,y,T). \tag{222}$$

Therefore, $\dfrac{\delta\hat{\rho}_1}{\delta T}$ can be written as

$$\frac{\delta\hat{\rho}_1}{\delta T} = i\gamma\left(\frac{\bar{\rho}_0}{\omega_0}\frac{\delta g}{\delta T} + \frac{g\partial_T\bar{\rho}_0}{\omega_0} - \frac{\bar{\rho}_0 g\partial_T\omega_0}{\omega_0^2}\right) - \bar{\rho}_0^2\frac{\delta\hat{\Theta}_1}{\delta T} - 2\bar{\rho}_0\partial_T\bar{\rho}_0\hat{\Theta}_1. \tag{223}$$

The term $\dfrac{\delta\hat{\Theta}_1}{\delta T}$ being canceled by a similar one in (208), we end up with only one functional derivative left in $\mathcal{H}_r$, that reads in the solvability condition

$$X = -\frac{\omega_0}{\gamma}\iint dxdyg(x)\times\frac{iA(T)}{\omega_0\bar{\rho}_0}\times\left(i\gamma\frac{\bar{\rho}_0}{\omega_0}\frac{\delta g}{\delta T}\right) = \frac{A(T)}{\omega_0}\iint dxdyg(x)\frac{\delta g}{\delta T}. \tag{224}$$

The integral over y is immediately computed, since nothing depends on y. Going back to more formal notations, we have

$$X = \frac{A(T)}{\omega_0} \int_0^{2\pi} dx g(x) \frac{\delta g_x}{\delta \alpha} (\partial_T \alpha) = \frac{A(T)}{2\omega_0} \int_0^{2\pi} dx \frac{\delta g_x^2}{\delta \alpha} (\partial_T \alpha). \tag{225}$$

We then use the formal definition of the functional derivative (215) together with the normalization condition (106):

$$X = \frac{A(T)}{2\omega_0} \lim_{\varepsilon \to 0} \left(\frac{\int_0^{2\pi} dx g_x^2 (\alpha + \varepsilon \partial_T \alpha) - \int_0^{2\pi} dx g_x^2(\alpha)}{\varepsilon} \right) = \frac{A(T)}{2\omega_0} \lim_{\varepsilon \to 0} \left(\frac{1-1}{\varepsilon} \right) = 0. \tag{226}$$

Finally, we end up with no functional derivative left from $\mathcal{H}_r$ in the solvability condition. Let us have a look at the other contribution, *i.e.* the term $\frac{\delta \hat{u}_1}{\delta T}$ in $\mathcal{F}_i$ (see (205)). It reads

$$Y = \iint dx dy \hat{u}_1 \times \omega_0 \bar{\rho}_0 A(T) \frac{\delta \hat{u}_1}{\delta T}. \tag{227}$$

According to (98), $\hat{u}_1 = -g'/(\omega_0^2 \bar{\rho}_0)$, and then

$$Y = \frac{A(T)}{\omega_0} \iint dx dy g'(x) \frac{\delta}{\delta T} \left(\frac{g'}{\omega_0^2 \bar{\rho}_0} \right). \tag{228}$$

As previously, we simplify the functional derivative,

$$\begin{aligned} Y =& \frac{A(T)}{\omega_0} \iint dx dy g'(x) \left(\frac{1}{\omega_0^2 \bar{\rho}_0} \frac{\delta g'}{\delta T} - \frac{2 g' \partial_T \omega_0}{\bar{\rho}_0 \omega_0^3} - \frac{g' \partial_T \bar{\rho}_0}{\omega_0^2 \bar{\rho}_0^2} \right) & (229) \\ =& \frac{A(T)}{\omega_0^3} \int dx \alpha g' \frac{\delta g'}{\delta T} - \frac{2A(T) \partial_T \omega_0}{\omega_0^4} \int dx \alpha g'^2 - \frac{A(T)}{\omega_0^3} \int dx g'^2(x) \int dy \frac{\partial_T \bar{\rho}_0}{\bar{\rho}_0^2}. & (230) \end{aligned}$$

The normalization condition (106) with the constitutive relation on g (100) gives

$$\int_0^{2\pi} dx g'(x)^2 \alpha(x) = \left[g(x) \alpha(x) g'(x) \right]_0^{2\pi} + \int_0^{2\pi} dx g(x) \times (\omega_0^2 g(x)) = \omega_0^2. \tag{231}$$

The second term of Y, involving $\partial_T \omega_0$, can then be immediately computed. The functional derivative becomes

$$\begin{aligned} \frac{A(T)}{\omega_0^3} \int dx \alpha g' \frac{\delta g'}{\delta T} =& \frac{A(T)}{2\omega_0^3} \int dx \alpha \frac{\delta g'^2}{\delta T} = \frac{A(T)}{2\omega_0^3} \int dx \left(\frac{\delta(\alpha g'^2)}{\delta T} - g'^2 \frac{\delta \alpha}{\delta T} \right) & (232) \\ =& \frac{A(T)}{2\omega_0^3} \left[\lim_{\varepsilon \to 0} \left(\frac{\int dx [\alpha g'^2][\alpha + \varepsilon \partial_T \alpha] - \int dx [\alpha g'^2][\alpha]}{\varepsilon} \right) - \int dx g'(x)^2 \partial_T \alpha \right] & (233) \\ =& \frac{A(T)}{2\omega_0^3} \left[2\omega_0 \partial_T \omega_0 - \int dx g'(x)^2 \partial_T \alpha \right] = \frac{A(T) \partial_T \omega_0}{\omega_0^2} - \frac{A(T)}{2\omega_0^3} \int dx g'(x)^2 \partial_T \alpha. & (234) \end{aligned}$$

Coming back to Y, we get,

$$Y = -\frac{A(T) \partial_T \omega_0}{\omega_0^2} - \frac{A(T)}{2\omega_0^3} \int dx g'(x)^2 \partial_T \alpha - \frac{A(T)}{\omega_0^3} \int dx g'^2(x) \int dy \frac{\partial_T \bar{\rho}_0}{\bar{\rho}_0^2}, \tag{235}$$

and finally, with the definition of α,

$$Y = -\frac{A(T)\partial_T\omega_0}{\omega_0^2} + \frac{A(T)}{2\omega_0^3}\int dx g'(x)^2\partial_T\alpha. \tag{236}$$

Again, the functional derivative is simplified with the use of the normalization condition: finally, we no longer have them in the solvability condition.

C.8 Step 4, part 3: simplifying the first part of the solvability condition

According to (207), the solvability condition can be written as

$$\underbrace{\iint dxdy\hat{u}_1\left(\mathcal{F}_i - \frac{\omega_0}{\gamma}\partial_x\mathcal{H}_r\right)}_{S_1} - \underbrace{\frac{\omega_0}{\gamma}\iint dxdyg(x)\mathcal{H}_r}_{S_2} = 0, \tag{237}$$

and we are in this section interested in simplifying S_1. With (205), we have

$$S_1 = \iint dxdy\hat{u}_1\omega_0\bar{\rho}_0\frac{dA}{dT}\hat{u}_1 - \iint dxdy\hat{u}_1\frac{\omega_0 A(T)}{Re_s h^2}\partial_{yy}\hat{u}_1 \tag{238}$$

$$+ A(T)\iint dxdy\hat{u}_1\omega_0\bar{\rho}_0\left(\frac{\delta\hat{u}_1}{\delta T} + \hat{u}_1\partial_x\bar{u}_1 + \bar{u}_1\partial_x\hat{u}_1 + \hat{v}_1\partial_y\bar{u}_1 + \bar{v}_1\partial_y\hat{u}_1\right). \tag{239}$$

We detail all these terms successively:

- Combining (231) and (98), the first term is found to be $\omega_0^{-1}\dfrac{dA}{dT}$.
- The second term can be written as

$$-\iint dxdy\hat{u}_1\frac{\omega_0 A(T)}{Re_s h^2}\partial_{yy}\hat{u}_1 = -\frac{A(T)}{Re_s h^2\omega_0^3}\int dx g'(x)^2\int dy\frac{1}{\bar{\rho}_0}\partial_{yy}\frac{1}{\bar{\rho}_0}. \tag{240}$$

  Integrating by part this integral splits it into a sign-definite bulk dissipation and a sign-indefinite power exchanges at the solid boundaries.

- The third term involves the functional derivative of $\hat{u}_1$ and has already been worked out, see (236). It reads

$$A(T)\omega_0\iint dxdy\hat{u}_1\bar{\rho}_0\frac{\delta\hat{u}_1}{\delta T} = -\frac{A(T)\partial_T\omega_0}{\omega_0^2} + \frac{A(T)}{2\omega_0^3}\int dx g'(x)^2\partial_T\alpha. \tag{241}$$

- The fourth term is:

$$A(T)\iint dxdy\hat{u}_1\omega_0\bar{\rho}_0\hat{u}_1\partial_x\bar{u}_1 = \frac{A(T)}{\omega_0^3}\iint dxdy\frac{g'(x)^2}{\bar{\rho}_0}\partial_x\bar{u}_1 \tag{242}$$

$$= -\frac{A(T)}{\omega_0^3}\iint dxdy\bar{u}_1\left(\frac{2g'(x)g''(x)}{\bar{\rho}_0} + g'(x)^2\partial_x\bar{\rho}_0^{-1}\right). \tag{243}$$

- The fifth term is

$$A(T)\iint dxdy\hat{u}_1\omega_0\bar{\rho}_0\bar{u}_1\partial_x\hat{u}_1 = \frac{A(T)}{\omega_0^3}\iint dxdyg'(x)\bar{u}_1\partial_x\left(\frac{g'(x)}{\bar{\rho}_0}\right) \tag{244}$$

$$= \frac{A(T)}{\omega_0^3}\iint dxdy\bar{u}_1\left(g'(x)^2\partial_x\bar{\rho}_0^{-1} + \frac{g'(x)g''(x)}{\bar{\rho}_0}\right). \tag{245}$$

- The sixth term is

$$A(T)\iint dxdy\hat{u}_1\omega_0\bar{\rho}_0\hat{v}_1\partial_y\bar{u}_1 = -\frac{A(T)}{\omega_0}\iint dxdyg'(x)\hat{v}_1\partial_y\bar{u}_1 \tag{246}$$

$$= \frac{A(T)}{\omega_0}\iint dxdyg'(x)\bar{u}_1\partial_y\hat{v}_1 \tag{247}$$

$$= \frac{A(T)}{\omega_0}\iint dxdyg'(x)\bar{u}_1\left(g(x) + \frac{g'(x)}{\omega_0^2}\partial_x\bar{\rho}_0^{-1} + \frac{g''(x)}{\omega_0^2\bar{\rho}_0}\right). \tag{248}$$

- The last term is

$$A(T)\iint dxdy\hat{u}_1\omega_0\bar{\rho}_0\bar{v}_1\partial_y\hat{u}_1 = \frac{A(T)}{\omega_0^3}\iint dxdyg'(x)^2\bar{v}_1\partial_y\bar{\rho}_0^{-1}. \tag{249}$$

Putting all this together, we get

$$S_1 = \frac{1}{\omega_0}\frac{dA}{dT} - \frac{A(T)}{Re_s h^2\omega_0^3}\iint dxdy\frac{g'(x)^2}{\bar{\rho}_0}\partial_{yy}\bar{\rho}_0^{-1} - \frac{A(T)\partial_T\omega_0}{\omega_0^2} + \frac{A(T)}{2\omega_0^3}\int dxg'(x)^2\partial_T\alpha \tag{250}$$

$$+ \frac{A(T)}{\omega_0^3}\iint dxdy\bar{u}_1\left(g'(x)^2\partial_x\bar{\rho}_0^{-1} + \omega_0^2 g(x)g'(x)\right) + \frac{A(T)}{\omega_0^3}\iint dxdyg'(x)^2\bar{v}_1\partial_y\bar{\rho}_0^{-1}.$$

C.9 Step 4, part 4: simplifying the second part of the solvability condition

We are interested in simplifying S_2, that is defined as

$$S_2 = \frac{\omega_0}{\gamma}\iint dxdyg(x)\mathcal{H}_r, \tag{251}$$

with $\mathcal{H}_r$ given by (208) simplified with (209),

$$\mathcal{H}_r = -\frac{\gamma g(x)}{\omega_0^2}\frac{dA}{dT} + \frac{i}{\omega_0\bar{\rho}_0}A(T)\left[\frac{\delta\hat{\rho}_1}{\delta T} + \partial_x(\hat{\rho}_1\bar{u}_1) + \partial_y(\hat{\rho}_1\bar{v}_1)\right] \tag{252}$$

$$+ \frac{iA(T)\bar{\rho}_0}{\omega_0}\left(\frac{\delta\hat{\Theta}_1}{\delta T} + \bar{u}_1\partial_x\hat{\Theta}_1 + \bar{v}_1\partial_y\hat{\Theta}_1\right)$$

$$+ \frac{iA(T)\bar{\rho}_0}{\omega_0}(\gamma - 1)\left[\hat{\Theta}_1(\partial_x\bar{u}_1 + \partial_y\bar{v}_1)\right] - \frac{iA(T)}{\omega_0}\frac{\gamma}{Pe_s h^2}\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right).$$

Again, we compute this term by term:

- The first term is

$$\frac{\omega_0}{\gamma}\iint dxdyg(x)\times\left(-\frac{\gamma g(x)}{\omega_0^2}\frac{dA}{dT}\right) = -\frac{1}{\omega_0}\frac{dA}{dT}. \tag{253}$$

- The two functional derivatives give, according to (223),

$$\frac{\omega_0}{\gamma}\iint dxdyg(x)\times\left(\frac{i}{\omega_0\bar{\rho}_0}A(T)\frac{\delta\hat{\rho}_1}{\delta T}+\frac{iA(T)\bar{\rho}_0}{\omega_0}\frac{\delta\hat{\Theta}_1}{\delta T}\right) \tag{254}$$

$$=\frac{iA(T)}{\gamma}\iint dxdy\frac{g(x)}{\bar{\rho}_0}\times\left(-2\bar{\rho}_0\partial_T\bar{\rho}_0\hat{\Theta}_1+i\gamma\left(\frac{\bar{\rho}_0}{\omega_0}\frac{\delta g}{\delta T}+\frac{g\partial_T\bar{\rho}_0}{\omega_0}-\frac{\bar{\rho}_0 g\partial_T\omega_0}{\omega_0^2}\right)\right)$$

$$=-\frac{2iA(T)}{\gamma}\iint dxdyg(x)\hat{\Theta}_1\partial_T\bar{\rho}_0+\frac{A(T)}{\omega_0}\iint dxdyg(x)^2\left(\frac{\partial_T\omega_0}{\omega_0}-\frac{\partial_T\bar{\rho}_0}{\bar{\rho}_0}\right) \tag{255}$$

$$=-\frac{2iA(T)}{\gamma}\iint dxdyg(x)\hat{\Theta}_1\partial_T\bar{\rho}_0-\frac{A(T)}{\omega_0}\iint dxdyg(x)^2\frac{\partial_T\bar{\rho}_0}{\bar{\rho}_0}+\frac{A(T)\partial_T\omega_0}{\omega_0^2}. \tag{256}$$

- The term involving $\hat{\rho}_1$ is,

$$\frac{\omega_0}{\gamma}\iint dxdyg(x)\times\left(\frac{i}{\omega_0\bar{\rho}_0}A(T)(\partial_x(\hat{\rho}_1\bar{u}_1)+\partial_y(\hat{\rho}_1\bar{v}_1))\right) \tag{257}$$

$$=-\frac{iA}{\gamma}\iint dxdy\left(-\bar{\rho}_0^2\hat{\Theta}_1+\frac{i\gamma\bar{\rho}_0 g}{\omega_0}\right)\times\left(\bar{u}_1\partial_x(\frac{g}{\bar{\rho}_0})+\bar{v}_1\partial_y(\frac{g}{\bar{\rho}_0})\right) \tag{258}$$

$$=\frac{iA}{\gamma}\iint dxdy\hat{\Theta}_1\left(\bar{\rho}_0 g'\bar{u}_1-g\bar{u}_1\partial_x\bar{\rho}_0-g\bar{v}_1\partial_y\bar{\rho}_0\right) \tag{259}$$

$$+\frac{A}{\omega_0}\iint dxdyg(x)\bar{\rho}_0\left(\bar{u}_1\frac{g'(x)}{\bar{\rho}_0}+\bar{u}_1 g(x)\partial_x\bar{\rho}_0^{-1}+\bar{v}_1 g(x)\partial_y\bar{\rho}_0^{-1}\right). \tag{260}$$

- Other terms read, with the use of (60),

$$\frac{\omega_0}{\gamma}\iint dxdyg(x)\times\frac{i\bar{\rho}_0}{\omega_0}A(T)\left(\bar{u}_1\partial_x\hat{\Theta}_1+\bar{v}_1\partial_y\hat{\Theta}_1\right) \tag{261}$$

$$=\frac{iA(T)}{\gamma}\iint dxdyg(x)\bar{\rho}_0\left(\bar{u}_1\partial_x\hat{\Theta}_1+\bar{v}_1\partial_y\hat{\Theta}_1\right) \tag{262}$$

$$=-\frac{iA(T)}{\gamma}\iint dxdy\hat{\Theta}_1\left(g'(x)\bar{\rho}_0\bar{u}_1+g(x)\partial_x\left[\bar{\rho}_0\bar{u}_1\right]+g(x)\partial_y\left[\bar{\rho}_0\bar{v}_1\right]\right) \tag{263}$$

$$=\frac{iA(T)}{\gamma}\iint dxdy\hat{\Theta}_1\left(g(x)\partial_T\bar{\rho}_0-g'(x)\bar{\rho}_0\bar{u}_1\right). \tag{264}$$

- We also have

$$\frac{\omega_0}{\gamma}\iint dxdyg(x)\times\frac{i\bar{\rho}_0}{\omega_0}A(T)\left((\gamma-1)\hat{\Theta}_1(\partial_x\bar{u}_1+\partial_y\bar{v}_1)\right) \tag{265}$$

$$=(\gamma-1)\frac{iA(T)}{\gamma}\iint dxdy\hat{\Theta}_1 g(x)\bar{\rho}_0(\partial_x\bar{u}_1+\partial_y\bar{v}_1) \tag{266}$$

$$=(1-\gamma)\frac{iA(T)}{\gamma}\iint dxdy\hat{\Theta}_1 g(x)\left(\partial_T\bar{\rho}_0+\bar{u}_1\partial_x\bar{\rho}_0+\bar{v}_1\partial_y\bar{\rho}_0\right). \tag{267}$$

- Finally, the thermal diffusion term reads

$$\frac{\omega_0}{\gamma}\iint dxdyg(x)\times-\frac{iA(T)}{\omega_0}\frac{\gamma}{Pe_s h^2}\left(\partial_{yy}\hat{\Theta}_1-\frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right) \tag{268}$$

$$=-\frac{iA(T)}{Pe_s h^2}\iint dxdyg(x)\left(\partial_{yy}\hat{\Theta}_1-\frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right). \tag{269}$$

All these results provide an expression for S_2:

$$S_2 = -\frac{1}{\omega_0}\frac{dA}{dT} + \frac{A(T)\partial_T\omega_0}{\omega_0^2} - \frac{iA(T)}{Pe_s h^2}\iint dxdy g(x)\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right) \tag{270}$$

$$+ \frac{A}{\omega_0}\iint dxdy g(x)\bar{\rho}_0\left(\bar{u}_1\frac{g'(x)}{\bar{\rho}_0} + \bar{u}_1 g(x)\partial_x\bar{\rho}_0^{-1} + \bar{v}_1 g(x)\partial_y\bar{\rho}_0^{-1}\right) \tag{271}$$

$$- \frac{A}{\omega_0}\iint g(x)^2\frac{\partial_T\bar{\rho}_0}{\bar{\rho}_0} - iA\iint \hat{\Theta}_1 g(x)\left(\partial_T\bar{\rho}_0 + \bar{u}_1\partial_x\bar{\rho}_0 + \bar{v}_1\partial_y\bar{\rho}_0\right). \tag{272}$$

This can be simplified with the continuity equation,

$$S_2 = -\frac{1}{\omega_0}\frac{dA}{dT} + \frac{A(T)\partial_T\omega_0}{\omega_0^2} - \frac{iA(T)}{Pe_s h^2}\iint dxdy g(x)\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right) \tag{273}$$

$$+ \frac{A}{\omega_0}\iint dxdy\left(g(x)g'(x)\bar{u}_1 + g(x)^2(\partial_x\bar{u}_1 + \partial_y\bar{v}_1)\right) \tag{274}$$

$$+ iA\iint \hat{\Theta}_1 g(x)\bar{\rho}_0\left(\partial_x\bar{u}_1 + \partial_y\bar{v}_1\right). \tag{275}$$

C.10 Step 4, part 5: A final expression for the solvability condition

The solvability condition $S_1 = S_2$ can finally be expressed, and is after simplification

$$\frac{2}{A}\frac{\mathrm{d}A}{\mathrm{d}T} - \frac{2}{\omega_0}\frac{\mathrm{d}\omega_0}{\mathrm{d}T} = -\frac{i\omega_0}{Pe_s h^2}\iint dxdy g(x)\left(\partial_{yy}\hat{\Theta}_1 - \frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right) \tag{276}$$

$$+ \iint dxdy g(x)^2(\partial_x\bar{u}_1 + \partial_y\bar{v}_1) + i\omega_0\iint \hat{\Theta}_1 g(x)\bar{\rho}_0\left(\partial_x\bar{u}_1 + \partial_y\bar{v}_1\right) \tag{277}$$

$$+ \frac{1}{Re_s h^2\omega_0^2}\iint dxdy\frac{g'(x)^2}{\bar{\rho}_0}\partial_{yy}\bar{\rho}_0^{-1} - \frac{1}{2\omega_0^2}\int dx g'(x)^2\partial_T\alpha \tag{278}$$

$$- \frac{1}{\omega_0^2}\iint dxdy g'(x)^2\left(\bar{u}_1\partial_x\bar{\rho}_0^{-1} + \bar{v}_1\partial_y\bar{\rho}_0^{-1}\right).$$

This can still be simplified:

- Note that

$$-\frac{1}{2\omega_0^2}\int dx g'(x)^2\partial_T\alpha = \frac{1}{2\omega_0^2}\iint dxdy g'(x)^2\frac{\partial_T\bar{\rho}_0}{\bar{\rho}_0^2} \tag{279}$$

$$= -\frac{1}{2\omega_0^2}\iint dxdy g'(x)^2\left(\frac{\partial_x\bar{u}_1 + \partial_y\bar{v}_1}{\bar{\rho}_0} - \bar{u}_1\partial_x\bar{\rho}_0^{-1} - \bar{v}_1\partial_y\bar{\rho}_0^{-1}\right). \tag{280}$$

- Moreover,

$$\frac{i\omega_0}{Pe_s h^2}\iint dxdy g(x)\left(\frac{\hat{\rho}_1}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0\right) = i\omega_0\iint dxdy\left(\frac{i\gamma g}{\omega_0} - \bar{\rho}_0\hat{\Theta}_1\right)\times\left(\partial_x\bar{u}_1 + \partial_y\bar{v}_1\right). \tag{281}$$

- The viscous term can also be written, since $\partial_{yy}\bar{\rho}_0^{-1} = \partial_{yy}\bar{\Theta}_0$,

$$\iint dxdy\frac{g'(x)^2}{\bar{\rho}_0}\partial_{yy}\bar{\Theta}_0 = Pe_s h^2\iint dxdy\frac{g'(x)^2}{\bar{\rho}_0}(\partial_x\bar{u}_1 + \partial_y\bar{v}_1). \tag{282}$$

Therefore, we obtain

$$\frac{2}{A\omega_0^{-1}}\frac{\mathrm{d}(A\omega_0^{-1})}{\mathrm{d}T} = -\frac{i\omega_0}{Pe_s h^2}\iint dxdy g(x)\partial_{yy}\hat{\Theta}_1 \tag{283}$$

$$+\iint dxdy(\partial_x\bar{u}_1+\partial_y\bar{v}_1)\left[(1-\gamma)g(x)^2+\frac{g'(x)^2}{\omega_0^2\bar{\rho}_0}\left(\frac{Pe_s}{Re_s}-\frac{1}{2}\right)\right] \tag{284}$$

$$-\frac{1}{2\omega_0^2}\iint dxdy g'(x)^2\left(\bar{u}_1\partial_x\bar{\rho}_0^{-1}+\bar{v}_1\partial_y\bar{\rho}_0^{-1}\right) \tag{285}$$

D Function G for any Γ

The solution of (135) for any value of Γ is obtained with Mathematica,

$$G_\Gamma(y) = \frac{A+B+C}{D}, \tag{286}$$

where

$$A = \Gamma^2(y-1)y\left[45+\Gamma\left(46+3\Gamma+y+3\Gamma y-(9+7\Gamma)y^2+3(2+\Gamma)y^3\right)\right]-30\Gamma\ln(1+\Gamma y),$$

$$B = 3\Gamma^2(\Gamma+1)y\left(-7+10y-5y^3+2y^4\right)\ln(\Gamma)+30(1+\Gamma)(1+2\Gamma)y\ln(1+\Gamma)-30\ln(1+\Gamma y),$$

$$C = -3\Gamma(1+\Gamma)y\left(\Gamma(-7+y^3(-5+2y))\ln\left(\Gamma(1+\Gamma)\right)+20\ln(1+\Gamma y)+10\Gamma y\ln\left(\Gamma(1+\Gamma y)\right)\right),$$

and

$$D = 1080\Gamma^3\left[\Gamma(2+\Gamma)+2(1+\Gamma)\ln(\Gamma)-2(1+\Gamma)\ln\left(\Gamma(1+\Gamma)\right)\right].$$