WHOI Math Review

Linear Algebra

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Outline

- Vectors
- Vector spaces
- Matrices
- Matrix algebra
- Systems of eqns
- Matrix inverses
- How to compute a determinant
- Eigenvalue decompo

WHOI Math Review

Linear algebra

vector: list of values (as opposed to a scalar)

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Sum: $\underline{a} + \underline{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$

Scalar product:

$$k\underline{a} = \begin{bmatrix} k a_1 \\ k a_2 \end{bmatrix}$$

Linear combination of vectors:

$$k_1 \underline{a} + k_2 \underline{b} = \begin{bmatrix} k_1 a_1 + k_2 b_1 \\ k_1 a_2 + k_2 b_2 \end{bmatrix}$$

Vector inner product:

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 \qquad \text{(a scalar!)}$$

```
>> a = [1
        2]
>> b = [3
        4]
>> dot(a, b)
```

Vector norm:

$$\|a\| = \sqrt{a \cdot a} = \sqrt{a_1 a_1 + a_2 a_2}$$

```
>> norm(a)
```

Orthogonality: $\underline{a}, \underline{b}$ orthogonal if $a \cdot b = 0$

Unit vector

$\|\underline{a}\| = 1 \Rightarrow \underline{a}$ is a unit vector

$\hat{\underline{a}} = \dfrac{\underline{a}}{\|\underline{a}\|}$ "normalized" $(\|\hat{\underline{a}}\| = 1)$

>> norm

STOP

Show in MATLAB

Vectors live in vector spaces that are characterized by dimension, e.g. $\mathbb{R}^2$

snakes $\in \mathbb{R}^2$

An example: taste space. Different values describe different foods.

$$\underline{t} = \begin{bmatrix} \text{sweet} \\ \text{sour} \\ \text{bitter} \\ \text{salty} \\ \text{umami} \end{bmatrix} \in \mathbb{R}^5$$

The vector space is all tastes.

Each taste sensation is special in that no matter how much of it we have, it will not taste like another.

i.e. $\nexists\ k_1, k_2$ st

$$k_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

This is the definition of linear independence between vectors.

A <u>basis</u> is a set of vectors such that every thing in the vector space can be expressed as a linear combination of those vectors.

$$\begin{bmatrix} \text{any} \\ \text{food} \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \dots + k_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Matrices

$$\underline{t} \in \begin{bmatrix} \text{sweet} \\ \text{sour} \\ \text{bitter} \end{bmatrix}$$

Imagine you have taken a mind-altering drug.

Say that elements of $\underline{m}$ are related to $\underline{t}$ by

$$m_1 = 2t_1 - t_2 + 0\,t_3$$
$$m_2 = -t_1 + 2t_2 - t_3$$
$$m_3 = 0t_1 - t_2 + 2t_3$$

This transformation can be written as a matrix operation,

$$\underline{m} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \underline{t}$$

Call this matrix $\underline{A}$ (upper case)

$$\underline{m} = \underline{A}\,\underline{t}$$

$\underline{A}$ is an example of a linear vector-valued function

To get $\underline{m}$ we have to perf matrix mult on $\underline{t}$.

3 rows & 3 columns $\rightarrow$ 3×3 matrix

The element in the i^{th} row & j^{th} location is a_{ij}

In general,

$C = AB$	$C_{ij} = \sum_k a_{ik} b_{jk}$	$C_{ij} = A_{ik} B_{kj}$
(vector)	(index)	(Einstein summation)

Systems of linear eqns

A system of linear equations is a collection of linear eqns involving the same set of variables. e.g., from before

$$m_1 = 2t_1 - t_2 \qquad \ldots$$

Imagine now that we know what the $\underline{m}$ are, and we want to know what we are eating.

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = A\underline{t}$$

How can we find $\underline{t}$?

- substitution
- Gaussian elimination
- the matrix inverse!

Matrix inverses

$\underline{m} = \underline{A}\, \underline{t}$

How to get t back?

Need the matrix inverse! only for square matrices

$AA^{-1} = A^{-1}A = I$

in MATLAB:

```
>> inv(A)
```

or "matrix division"

```
>> A * (A \ eye(size(A)))
```

We can only do this if A is nonsingular.

One way we know: if its determinant is nonzero.

```
>> det (A)
```

$|A|$

For a 2x2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For larger matrices:

$$|A| = \sum_{i} 1^{i+j} a_{ij} \underbrace{m_{ij}}_{}$$

"minor" - calculated by taking the determinant of $|A|$ with i^{th} row & j^{th} column removed

e.g.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} =$$

$$aei - afh + bdi - bfg + cdh - ceg$$

Eigenvalue decompositions

Obey $A\underline{v} = \lambda \underline{v}$

Why? modes of the problem

raspberries still taste like raspberries

To find eigenvalues / vectors:

1) Compute eigenvalues

$$\det(A - \lambda I) = 0$$

e.g. $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$Bv = \lambda v$$

$(B - \lambda I)v = 0$ has a sol'n only when $|B - \lambda I| = 0$

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2 = 0 \qquad \text{(char poly)}$$

has roots $\lambda_1 = 1$, $\lambda_2 = 3$

2) Compute eigenvectors

$\lambda = 1$:

$$(A - I\lambda_1)v = (A - I)\underline{v} = \underline{0}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \end{matrix} \qquad \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 3$: $(A - 3I)w = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \qquad \underline{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

What's the point??