MIT-WHOI JP Summer Math Review
Notes from Class
Topic: **Partial Differential Equations**
Instructor: **Alec Bogdanoff** (alecb@mit.edu)
Based on Julia Hopkins's notes.

Reference text: Advanced Analytic Methods in Applied Mathematics, Science, and Engineering by Hung Cheng

Basic Partial Differential Equations (PDEs)

The difference between partial differential equations and the ordinary differential equations covered in previous review classes is simple: an ODE has one independent variable, a PDE has two or more. This seemingly straightforward difference necessitates an entirely different approach to solving PDEs than ODEs, and also leads many to believe that PDEs are intrinsically more difficult to solve than ODEs.

In many ways, this belief is correct; many PDEs cannot be solved. However, these notes aim to convince you that this complexity does not necessarily translate to more difficult solutions, when solutions are possible. You just need a different set of tools to solve PDEs, many of which are explained below.

First-Order PDEs

The simplest types of PDEs are first-order, i.e. equations without anything higher than a first derivative. These PDEs can always be solved in a closed (analytic) form, usually by turning them into a system of ODEs. The typical solution method for these equations is called the *method of characteristics*.

We begin with simple linear, homogenous PDEs to introduce the *method of characteristics*.

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = 0$$

We are looking for the solution $u(x,y)$. Let

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y) = u_x\Delta x + u_y \Delta y$$

by Taylor expansion. Note that we neglect all higher order terms (assume infinitesimally small Δx and Δy). From the original equation, we have for $b \neq 0$ that

$$u_y = -\frac{a}{b}u_x$$

which then gives $\Delta u = \left(\Delta x - \frac{a}{b}\Delta y\right)u_x$. Thus we see that if Δx and Δy are infinitesimally small, in particular if $\Delta x = \frac{a}{b}\Delta y$, or $\frac{dx}{a} = \frac{dy}{b}$, we have $\Delta u = 0$ and a constant solution to u. At which point, the original PDE becomes

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

which is an ODE independent of u. The solution to this ODE dictates a set of *characteristic curves* along which the solution to u is constant. Thus we can solve for u along these curves once the curves have been identified.

This method requires some form of initial condition, i.e. $u(0, y) = f(y)$ or $u(x, 0) = f(x)$, to find an expression for u. Once a condition has been identified, the solution to the original linear, homogenous PDE can be found either graphically (draw the characteristic curves, find the one that meets the condition on u) or analytically. The details of the latter, usually the more tractable in practice, are given below:

1. Obtain the characteristic curves of the PDE and express them in the form $f(x, y) = a$.
2. The general solution of a linear, homogenous first-order PDE is then $u(x, y) = F(f(x, y))$.
3. Using the initial condition, i.e. $u(x, 0) = f(x)$, we can find $f(x) = F(f(x, 0))$ which then can be solved for F.

Examples

$$u_t - xu_x = 0 \qquad\qquad u(x, 0) = e^{-x^2}$$

$$yu_x - xu_y = 0 \qquad\qquad u(x, 0) = x$$

The basics of this technique hold for more general (nonlinear, non-homogenous) first-order PDEs as well. A more complex general first-order PDE, however, requires us to modify the solution method. We look first at the non-homogeneous case where the function is linear in u_x and u_y, but not necessarily in u.

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

Thus we have, similar to the linear case:

$$\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)}$$

when we take Δx and Δy small enough. Since this is no longer dependent only on x and y, however, we must solve the above equation with the extra constraints from the non-homogenous equation

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)} = dt$$

where the variable t has been introduced for the sake of being able to rewrite the above in a form similar to the linear, homogenous cases:

$$\frac{dx}{dt} = a(x, y, u) \qquad \frac{dy}{dt} = b(x, y, u) \qquad \frac{du}{dt} = c(x, y, u)$$

and end up with a system of ODEs which can (ideally) be then solved for the characteristic curves, this time existing in the three-dimensional xyu-plane as opposed to the two-dimensional xy-plane we worked with in the linear, homogeneous case.

Examples

$$xu_x + yu_y = 1 + y^2 \qquad u(x, 1) = 1 + x$$

Consider a stream with fish swimming in a river current. Let's assume that the fish can be described by a density function $\rho(x,t)$ for position x and time t. Let the velocity of the fish relative to the water be $v(x,t)$. The flux of fish is then given by

$$q(x,t) = \rho(x,t)v(x,t)$$

With continuity (conservation of mass), we have

$$\frac{d\rho}{dt} + \frac{dq}{dx} = 0$$

Further, assume that the velocity of the fish can be modeled by $v = 1 - \rho$, such that a fish responds to the presence of other fish by slowing down from the ambient, normalized velocity of $v = 1$. This then gives

$$q = \rho(1 - \rho)$$

which, when plugged into the differential equation of continuity, gives

$$\rho_t + (1 - 2\rho)\rho_x = 0$$

Solve the above nonlinear first-order PDE with an initial fish density of $\rho(x,0) = \frac{1}{1+|x|}$

The above method of characteristic examples can be expanded to equations that are also nonlinear in u_x and u_y. The basic idea is that the dimensions of the space in which the characteristic curve exists becomes five-dimensional. The characteristic curves get more difficult to find and the initial conditions harder to apply. If you are interested in this topic, the reference used for this class has a good section on some of the subtleties of this particular type of first-order PDE.

The methods of characteristics as explained thus far (with nonlinearity in x, y and u) will solve a lot of the problems you are likely to encounter in the ocean sciences.

Higher Order PDEs

Our discussion of higher order PDEs acknowledges that there are many different ways to solve each particular type of PDE you will come across. The best preparation I can give you in an hour and a half is to be able to recognize some particular types of higher order PDEs and from there infer their solutions.

The number of solvable higher order PDEs are much fewer than the number of solvable first-order PDEs. For those which are able to be solved, however, the method of *Separation of Variables* is generally a good approach.

Separation of Variables

You will need a linear, homogenous partial differential equation and linear, homogenous boundary conditions. Caution: even when these requirements are met, this method may not work. It will, however, allow us to evaluate some basic PDEs fundamental to the sciences.

The method has one key step: we assume the solution to the PDE $u(x,t)$ is

$$u(x,t) = f(x)g(t)$$

This is purely a guess, and not one based on a lot of higher mathematical principles. It is a guess which exists simply because it seems to work more times than not with solvable PDEs.

Other than this assumption, the method involves applying this form of a solution to the PDEs in question by taking into account the boundary conditions and then working from there to find a solution. We will cover three critical examples in the sciences to illustrate this approach.

Heat Equation

The heat equation, also known as the diffusion equation, appears in several different physical applications. One of the most common in thermodynamics (which you might see very soon into your oceanographic career) is the problem of a heat source in a tank of water. The puzzle is to figure out how long it will take the heat to diffuse into the control volume of water, and the temperature at which the tank will reach equilibrium.

The heat equation has components in both time and space – we look at the 1D version below:

$$\frac{\partial \phi}{\partial t} = k\frac{\partial^2 \phi}{\partial x^2}$$

This equation can be derived from the equation of mass conservation if ϕ is thought to be density in a control volume. The equation effectively states that the density of particles in a control volume is equal to the movement of particles in and out of that control volume over time in absence of other sources/sinks in the CV. We will not go through the full derivation here – we will instead focus on how to solve the equation.

In order to find a unique solution to this equation, we need some boundary conditions and initial conditions. Since the heat equation has a first-order derivative in time, we will need one initial condition. Since it has a second-order derivative in space, we need two boundary conditions to close the problem.

We can prescribe a number of boundary conditions, which usually fall into four categories.

Dirichlet conditions: prescribe temperature (or particle density) values at the boundaries
Neumann conditions: prescribe the flux of heat (or particles) at the boundaries
Mixed conditions: prescribe a Dirichlet condition on one end and a Neumann on the other
Robin conditions: prescribe both Dirichlet and Neumann boundary conditions on a boundary

For the below examples of solving the heat equation using separation of variables, we will begin with simple Dirichlet boundaries

$$\phi(0,t) = 0 \qquad \phi(L,t) = 0 \qquad \phi(x,0) = f(x)$$

The solution for Neumann conditions has some small, but significant differences. The governing equation remains the same, but the boundary conditions become

$$\frac{d\phi}{dx}(0,t) = 0 \qquad \frac{d\phi}{dx}(L,t) = 0 \qquad \phi(x,0) = f(x)$$

We finally look at the heat equation with non-homogenous boundary conditions, a more realistic situation in physical systems

$$\phi(0,t) = T_1 \qquad \phi(L,t) = T_2 \qquad \phi(x,0) = f(x)$$

Laplace Equation

The Laplace equation appears frequently in fluid mechanics as the basic description of conservation of mass in a fluid with constant density (irrotational and incompressible). It is written below for the 2D case

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

where φ is the stream function (you'll learn more about this in an introductory fluid mechanics course. I highly recommend taking one!)

In order to solve this equation, we need four boundary conditions. Again, the number of boundary conditions required depends on the order of the derivatives in your PDE. Since the Laplace equation above consists of two second-order derivatives, we need four boundary conditions to solve it. Those conditions can come in a variety of forms.

One of the simplest BC cases is that of a rectangle:

$$\varphi(0,y) = g_1(y) \qquad \varphi(L,y) = g_2(y)$$
$$\varphi(x,0) = f_1(x) \qquad \varphi(x,H) = f_2(x)$$

Note that these boundary conditions, while linear, are not homogenous. This is going to make the solution a bit trickier to find with separation of variables

We then solve the Laplace equation when the boundary is not a rectangle. If it is instead a circle, for instance, the equation must be translated to cylindrical or polar coordinates to make applying the boundary conditions as simple as possible. An example is given below, using polar coordinates

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0$$

$$|\varphi(0,\theta)| < \infty \qquad \varphi(a,\theta) = f(\theta)$$

$$\varphi(-\pi,t) = \varphi(\pi,t) \qquad \frac{\partial\varphi}{\partial\theta}(-\pi,t) = \frac{\partial\varphi}{\partial\theta}(\pi,t)$$

Real world La Place Example

Wave Equation

The wave equation can be used to describe a number of physical phenomena. For instance, it can describe the motion of a string stretched between two points and then perturbed. It can be used to describe a simple water wave field in either the ocean (2D-3D cases) or a wave flume (1D case).

We first deal with the 1D version of the wave equation, which varies in space x and time t.

$$\frac{d^2\phi}{dt} = c^2\frac{d^2\phi}{dx^2}$$

where ϕ is the displacement of the wave and c is an arbitrary constant (physically considered to be wave celerity—or wave phase speed). In order to solve this, we again need to prescribe boundary conditions and initial conditions (sounds familiar, right?). For the current example, we will use prescribed positions of the wave at either boundary

$$\phi(0,t) = 0 \qquad\qquad \phi(L,t) = 0$$

and initial conditions

$$\phi(x,0) = f(x) \qquad\qquad \frac{d\phi}{dt}(x,0) = g(x)$$

Again, we require two boundary conditions because of the second derivative in space, and likewise we need two initial conditions (position and slope) as a result of having a second derivative in time. With this set-up, we are in a position to solve a general version of the wave equation.

MATLAB example of the Heat Equation