MIT-WHOI Joint Program Summer Math Review **Summer 2015**

Lecture 8: Ordinary Differential Equations

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Disclaimer: *These notes are for the purposes of this review only and are not intended for wide distribution. Credit is due to former instructors, including Jay Brett and Melissa Moulton. NPZ model code was provided by John Taylor and collaborators.*

8.1 Learning Objectives

At the end of this session, you will be able to do the following:

1. recognize and classify ordinary differential equations.
2. solve linear first-order ordinary differential equations.
3. solve constant-coefficient linear second-order differential equations.
4. understand the basic functioning of an NPZ model

8.2 Definitions

Ordinary Differential Equation (ODE): an equation containing a function of one independent variable and its derivatives.

The general definition for an ODE is

$$x^{(n)} = F(t, x, x', ..., x^{(n-1)})$$

Where F is a given function of x,t, and derivatives of x. $x^{(n)}$ is the n-th derivative of the function x. This is an explicit ODE of **order** n.

F can also be given implicitly as follows:

$$F(t, x, x', ..., x^{(n)}) = 0$$

8.2.1 Example 1

A classic example of an ordinary differential equation is Newton's second law of motion $F = ma$.

Which can be written as: $m\frac{d^2x(t)}{dt^2} = F(x(t))$. Eqivalently: $F = mx''(t)$, $F = mx^{(2)}(t)$, $F = m\ddot{x}(t)$.

Here a, acceleration, is written as the second time derivative of position.

This is a second order ordinary differential equation where one aims to solve for x as a function of t.

8.2.2 More Definitions

Autonomous Differential Equation: A differential equation that doesn't depend on t (independent variable).

Linear DE: A differential equation is linear if F can be written as a linear combination of the derivatives of x.

Mathematically that looks like this:

$x^n = \sum_{k=0}^{n-1} a_k x^k + r(t)$

Where a_k is some coefficient and $r(t)$ is the source/sink term.

These linear differential equations have different names and solution methods depending on what your $r(t)$ is.

if $r(t) = 0$ then the equation is called **homogeneous**.

if $r(t) \neq 0$ then the equation is **nonhomogeneous**.

8.2.3 Classification Practice

Determine which of the following are ODEs. For each that is, note its order and determine whether it is any or all of: linear, autonomous, homogeneous, nonhomogeneous.

1. $$\frac{dy}{dt} + y = 0$$

2. $$y = x$$

3. $$\frac{dx}{dt} + x^2 = t$$

4. $$m\ddot{x} + b\dot{x} + kx = 0$$

5. $$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0$$

6. $$y' + t^2 y + t^3 = 0$$

8.3 Solving Differential Equations

8.3.1 Separable Differential Equations

There are several types of separable differential equations, but they all generally are solved by grouping t with functions and derivatives of t and doing the same for x.

Example 1. Here is an example of a differential equation that is separable in t:

$\frac{dx}{dt} = F(t)$

Where $F(t)$ is some function of t. Here is how this equation would be solved with $F(t) = t^2 + 5$:

$\frac{dx}{dt} = t^2 + 5$

$dx = (t^2 + 5)dt$

$\int dx = \int (t^2 + 5)dt$

After integrating you wind up with

$x = \frac{t^3}{3} + 5t + C$

Where C is an integration constant that requires an initial condition to resolve. This solution creates what's known as a **family of solutions** where different values of C create different solutions. An **initial condition** or **boundary condition** would result in a unique solution. From an initial condition, the statement $x(t = 0) = X$, you can find $C = X$. Boundary conditions are used when the independent variable represents space.

Another form of separable differential equations is:

$$P_1(t)Q_1(x) + P_2(t)Q_2(x)\frac{dx}{dt} = 0$$

Where $P_1(t)$, $P_2(t)$, $Q_1(x)$, and $Q_2(x)$ are functions of t and x respectively.

To solve this form of separable ODE you first divide through by P_2Q_1 and then separate and integrate.

Example 2.

Let's take $P_1 = t^2$, $P_2 = t^4$, $Q_1 = x$, and $Q_2 = x^4$

The equation now looks like this:

$$t^2x + t^4x^4\frac{dx}{dt} = 0$$

$P_2Q_1 = t^4x$ so dividing through gives

$$\frac{1}{t^2} + x^3\frac{dx}{dt} = 0.$$

Separating (by moving one term to the other side and multiplying by dt) gives

$x^3dx = -\frac{dt}{t^2}$.

Then we integrate:

$\int x^3 dx = \int -\frac{dt}{t^2}$, which gives

$\frac{x^4}{4} = \frac{1}{t} + C.$

Which again without an initial value gives you a family of solutions. Here, the 'initial value' must be from some time other than zero, as the solution is not defined there.

8.3.1.1 Practice

Solve:

1.
$$\frac{dy}{dt} + y = 0$$
$$y(0) = 5.$$

2.
$$tx' = t^2 x^{-1}$$
$$x(1) = 1.$$

8.3.2 The Integrating Factor

The integrating factor solution to first order, linear, nonhomogeneous ODEs with function coefficients is a popular solution taught in most differential equations courses and comes up surprisingly often.

The differential equation must be in the form or be able to be put into the form:

$$\frac{dx}{dt} + P(t)x = Q(t)$$

To solve this you apply a formula involving something called the integrating factor. The integrating factor is

$$e^{\int P(t)dt}.$$

Multiplying the equation by this factor gives:

$$e^{\int P(t)dt}\frac{dx}{dt} + P(t)e^{\int P(t)dt}x = e^{\int P(t)dt}Q(t) = \frac{d}{dt}(e^{\int P(t)dt}x),$$

by recognizing the chain rule. Then the equation can be simply integrated. The general solution looks like:

$$x = e^{-\int P(t)dt}\int Q(t)e^{\int P(t)dt}dt + Ce^{-\int P(t)dt}$$

Example:

$P(t) = \frac{2}{t}$, $Q(t) = 5t$

$$\frac{dx}{dt} + \frac{2}{t}x = 5t$$

Your integrating factor is then $e^{\int \frac{2}{t}dt}$ which works out to t^2.

Plugging into the general solution you get:

$$x = \frac{1}{t^2}\int 5t^3 dt + \frac{C}{t^2}.$$

Finally, after evaluating the integral you get

$$x = \frac{5t^2}{4} + \frac{C}{t^2}.$$

Note: if $Q(t) = 0$ then the equation is homogeneous and the solution is simply

$$x = \frac{C}{e^{\int P(t)dt}}$$

8.3.2.1 Practice

Solve:

1.

$$\frac{dy}{dt} + t^2 y = t^2$$
$$y(0) = 5.$$

2.

$$tx' = x + t$$
$$x(1) = 1.$$

8.3.3 Solving Second Order Homogeneous and Nonhomogeneous ODEs

This method works for linear second-order nonhomogeneous linear differential equations with constant coefficients. Along the way you solve for the solution to the homogeneous equation, called the **complimentary function**. The equation to be solved will be of the form

$$ax'' + bx' + cx = G(t)$$

and the complimentary equation is

$$ax'' + bx' + cx = 0.$$

For equations of this form the general solution can be stated as follows:

$$x(t) = x_p(t) + x_c(t)$$

Where x_p is the particular solution to the nonhomogeneous equation and x_c is the solution to the complementary equation. This solution method depends on the **superposition principle**, which states that for linear functions, the output of a sum of inputs is equal to the sum of outputs of those inputs: $f(a + b) = f(a) + f(b)$. (There's a proof in Stewart Calculus if you're into that kind of thing.)

8.3.3.1 Auxiliary Equation

To solve the complementary equation, which is 2nd-order, homogeneous, and linear with constant coefficients, we form the **auxiliary** or **characteristic equation**. This equation allows us to find a solution to the complementary equation as the sum of two exponentials. All second-order equations must have two solutions, or one with two undetermined coefficients, to allow initial conditions on the function and it's first derivative.

Example Let the complimentary equation we are solving be

$$mx'' + bx' + kx = 0.$$

Then suppose $x = Ae^{rt}$, and plug that in:

$$mr^2 Ae^{rt} + brAe^{rt} + kAe^{rt} = 0,$$
$$mr^2 + br + k = 0$$

where we have divided by x, is the characteristic equation. It is quadratic in r. The solutions give r_1 and r_2, and the full solution is then:

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$

where C_1 and C_2 would be determined by initial conditions.

The possibilities for the complementary equation are as follows:

r_1,r_2 are real and distinct, then the general solution is $x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$r_1 = r_2 = r$ then $x = C_1 e^{rt} + C_2 te^{rt}$

r_1, r_2 are complex $(\alpha \pm i\beta)$ then:

$x = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

8.3.3.2 Method of Undetermined Coeffiecients

The first method we'll look at is the method of undetermined coefficients. In this method you make the assumption that the particular solution is a polynomial of the same degree as $G(t)$.

Here's an example to show what the process looks like: **Example**

$$x'' + x' - 2x = t^2$$

So we'll start by solving the complementary equation:

$$r^2 + r - 2 = (r-1)(r+2) = 0$$

This gives us a $x_c = c_1 e^t + c_2 e^{-2t}$

Now we look for x_p of the form:

$$x_p = At^2 + Bt + C$$

(Because $G(t)$ is a second degree polynomial)

With x_p of that form, $x_p' = 2At + B$ and $x + p'' = 2A$. Now substitute these into the solution and solve for the coefficients A, B, and C.

$$(2A) + (2At + B) - 2(At^2 + Bt + C) = t^2 - 2At^2 + (2A - 2B)t + (2A + B - 2C) = t^2$$

Now you solve the system to make the coefficients equal to $G(x)$

$-2A = 1$

$2A - 2B = 0$

$2A + B - 2C = 0$

This gives us $A = \frac{-1}{2}$, $B = \frac{-1}{2}$, and $C = \frac{-3}{4}$

Putting it all together:

$$x = x_c + x_p = c_1 e^t + c_2 e^{-2t} - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}$$

Note: If $G(t)$ is an exponential you guess $x_p(t) = Ae^{kt}$; if G is a trigonometric function, you guess the sum of that function and a derivative or two, etc. However, for trigonometric forms, the next method is often better.

8.3.3.3 Method of Variation of Parameters

In this method you start off by taking the general solution $x(t) = C_1 x_1(t) + C_2 x_2(t)$ and replace the constants by arbitrary functions $u_1(t)$ and $u_2(t)$.

Now we look for a particular solution that looks as follows:

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$$

Now you differentiate the equation above and wind up with:

$$x_p' = (u_1' x_1 + u_2' x_2) + (u_1 x_1' + u_2 x_2')$$

Because these functions are arbitrary we can impose conditions on them to make our lives easier.

The condition commonly imposed is:

$$u_1'x_1 + u_2'x_2 = 0$$

It follows that:

$$x_p'' = u_1'x_1' + u_2'x_2' + u_1x_1'' + u_2x_2''$$

Then you can substitute all that into the differential equation. (We'll skip that because you'll see in the example)

Because both x_1 and x_2 are solutions to the complementary equation the end result is that

$$a(u_1'x_1' + u_2'x_2') = G$$

Example Solve the equation $x'' + x = \tan(t)$ from $0 < t < \frac{\pi}{2}$

The auxiliary equation is $r^2 + 1 = 0$

Which has the roots $r = \pm i$

So using the variation of parameters method we want a solution of the form:

$$x_p = u_1(t)\sin(t) + u_2(t)\cos(t)$$

Taking the derivative you wind up with:

$$x_p' = (u_1'\sin(t) + u_2'\cos(t)) + (u_1\cos(t) - u_2\sin(t))$$

Then we impose our condition:

$$u_1'\sin(t) + u_2'\cos(t) = 0$$

Then taking the second derivative:

$$x_p'' = u_1'\cos(t) - u_2'\sin(t) - u_1\sin(t) - u_2\cos(t)$$

Then adding x_p'' and x_p together (because the original equation was $x'' + x = tan(t)$) you wind up with:

$$u_1'\cos(t) - u_2'\sin(t) = \tan(t)$$

Our condition gives us:

$$u_2' = u_1'\frac{\sin(t)}{\cos(t)}$$

Substituting this into the x_p equation yields:

$$u_1'(\sin^2(t) + \cos^2(t)) = \cos(t)\tan(t)$$

Which gives: $u_1' = \sin t$ and $u_1 = -\cos t$

Plugging back into u_2' you get:

$$u_2' = -\frac{\sin t}{\cos t}u_1' = -\frac{\sin^2 t}{\cos t} = \frac{\cos^2 t - 1}{\cos t} = \cos t - \sec t$$

Then $u_2 = \sin t - \ln(\sec t + \tan t)$

Note that $\sec t + \tan t$ is positive from $0 < t < \frac{\pi}{2}$

so the particular solution is:

$$\begin{aligned} x_p &= -\cos(t)\sin(t) + [\sin(t) - \ln(\sec(t) + \tan(t))]\cos(t) \\ &= -\cos(t)\ln(\sec(t) + \tan(t)) \end{aligned}$$

and the general solution is:

$$x(t) = C_1 \sin(t) + C_2 \cos(t) - \cos(t)\ln(\sec(t) + \tan(t))$$

8.3.3.4 Practice

Solve using the method of undetermined coefficients:

$$x'' - x' + 2x = t^3.$$

Set up using variation of parameters:

$$x'' - x' + 2x = \cos(t).$$

8.4 NPZ model

NPZ (Nutrient, Phytoplankton, Zooplankton) models are used to study marine ecosystems in a simple way. Note that this is a system of ODEs. You will learn about how to solve these in the next lecture (though probably not ones this complicated). For now we will be getting a feel for this system and solving it numerically using MATLAB.

The basic ecosystem that this system of equations represents is one in which phytoplankton consumes nutrients, is eaten by zooplankton and die at a regular rate. When a phytoplankton dies, its "organic material" is recycled back into nutrients. Zooplankton grow by eating phytoplankton and also die off at a regular rate. The equations that have been found to represent this system are

$$\frac{dN}{dt} = -\mu P(\frac{N}{N + N_s}) + rP \tag{8.1}$$

$$\frac{dP}{dt} = \mu P(\frac{N}{N + N_s}) - \alpha PZ - rP \tag{8.2}$$

$$\frac{dZ}{dt} = \alpha\beta PZ - mZ \tag{8.3}$$

N,P and Z represent the amount of nutrients, phytoplankton and zooplankton in the system respectively. The parameters (that we will be changing in the model) are:

- N_s, saturation value of nutrients. Phytoplankton do not grow well if the available nutrients are much less than this.
- r, death rate of phytoplankton.
- m, death rate of zooplankton.
- α, rate of zooplanknton grazing of phytoplankton.
- β, efficiency of zooplankton conversion of phytoplankton biomass into zooplankton biomass.

We will use the scripts NPZ.m and NPZ_fun.m to solve these equations in MATLAB for different sets of initial conditions and parameters and learn about how this system works.

A few things to think about:

- Can you describe what is happening to each of the variables (NPZ) over time?
- How is this affected by changing the parameters?
- What happens when you introduce a seasonal/diurnal cycle in μ? (commented out in the script)
- What are the conditions for steady state for each of the parameters?

8.5 Some Useful Resources

Wikipedia ‘Ordinary Differential Equations’ – ’Integrating Factor’

Stewart Calculus (My favourite calculus books. Pretty much every edition I’ve seen is good)

Hibbeler (More so for Engineers)

Kundu-Cohen (Sort Of) – Some other fluid mechanics books do a better job at explanations but MIT seems to like this one.