MIT-WHOI Joint Program Summer Math Review **Summer 2015**

Lecture 12: Scaling and Nondemensionalization

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12.1 Learning Objectives

At the end of this session, you will be able to:

1. Scale variables to create nondimensional versions.
2. Nondimensionalize differential equations.
3. Simplify nondimensional equations to form nondimensional numbers.
4. Recognize common oceanographic nondimensional numbers.

12.2 Basic Principles

Nondimensionalization and scaling are useful tools for analyzing the behaviour of equations and determining which dynamics are important.

We wil be dealing with equations that have variables whose dynamics we are interested in describing (i.e. velocity, fish population) and constant parameters that affect the scale of the variables of interest (i.e. length and time scales). By *scaling* these variables and parameters using representative values so that they vary between zero and (order) one, each term in the equation of interest becomes a constant multiplied by a function of variables which is order one. Dividing through by one of those constants gives a *nondimensional* equation. This ensures that the relative size of different terms is clear, making solving the system at different limits straightforward. (You can then re-dimensionalize to get a physical result.)

We will explore these concepts through a variety of examples designed to give you a flavor for some different applications.

Wikipedia is a surprisingly good resource to remind yourself how to nondimensionalize:
http://en.wikipedia.org/wiki/Nondimensionalization.

Nondimensionalization steps [edit]

To nondimensionalize a system of equations, one must do the following:

1. Identify all the independent and dependent variables;
2. Replace each of them with a quantity scaled relative to a characteristic unit of measure to be determined;
3. Divide through by the coefficient of the highest order polynomial or derivative term;
4. Choose judiciously the definition of the characteristic unit for each variable so that the coefficients of as many terms as possible become 1;
5. Rewrite the system of equations in terms of their new dimensionless quantities.

The last three steps are usually specific to the problem where nondimensionalization is applied. However, almost all systems require the first two steps to be performed

As an illustrative example, consider a first order differential equation with constant coefficients:

$$a\frac{dx}{dt} + bx = Af(t).$$

1. In this equation the independent variable here is *t*, and the dependent variable is *x*.
2. Set $x = \chi x_c, \ t = \tau t_c$. This results in the equation
$$a\frac{x_c}{t_c}\frac{d\chi}{d\tau} + bx_c\chi = Af(\tau t_c) \stackrel{\text{def}}{=} AF(\tau).$$
3. The coefficient of the highest ordered term is in front of the first derivative term. Dividing by this gives
$$\frac{d\chi}{d\tau} + \frac{bt_c}{a}\chi = \frac{At_c}{ax_c}F(\tau).$$
4. The coefficient in front of χ only contains one characteristic variable t_c, hence it is easiest to choose to set this to unity first:
$\frac{bt_c}{a} = 1 \Rightarrow t_c = \frac{a}{b}$. Subsequently, $\frac{At_c}{ax_c} = \frac{A}{bx_c} = 1 \Rightarrow x_c = \frac{A}{b}$.
5. The final dimensionless equation in this case becomes completely independent of any parameters with units:
$$\frac{d\chi}{d\tau} + \chi = F(\tau).$$

12.3 Nondimensionalizing sets of equations

Consider a 1D diffusion model for momentum with constant density and linear drag. The governing equation is

$$\kappa\frac{\partial^2 u}{\partial z^2} = cu.$$

At the top there is a flux boundary condition that represents forcing by the wind

$$\kappa\frac{\partial u}{\partial z} = \frac{\tau}{\rho}.$$

Let the bottom boundary condition be u=0, so we will not worry about it here.

To nondimensionalize, I introduce non-dimensional variables, $u*$ and $z*$ related to u and z thus:

$$u = \sqrt{\tau/\rho}u* \qquad z = Hz*$$

where H is the dimensional depth. I chose these scales because the depth is the only physical length scale and the forcing at the surface will create the largest velocity, so the velocity scale must use τ as a measure of force and ρ as a measure of mass. Now I can rewrite my flux boundary condition and governing equations in terms of non-dimensional parameters:

$$\frac{\partial u*}{\partial z} = \alpha \qquad \frac{\partial^2 u*}{\partial z*^2} = \omega^2 u*$$

where

$$\omega = H\sqrt{\frac{c}{k}} \qquad \alpha = \frac{H\sqrt{\tau/\rho}}{k}.$$

This tells us that the frequency of oscillation (or the decay rate) does not depend on the wind forcing-only the boundary condition does, which will set the amplitude of the solution; both depend on the water depth and the diffusivity; only the frequency (decay rate) depends on the drag. The solution to this equation is (assuming z is positive upward and zero at the surface and we want a finite velocity)

$$u* = \frac{\alpha}{\omega}e^{\omega z}.$$

To redimensionalize, use the above defined relationship between nondimensional and dimensional variables.

12.4 Navier-Stokes

The Navier-Stokes equations are the momentum (conservation) equations for a fluid. Here they are:

$$\frac{\partial u}{\partial t} + \vec{u}\cdot\nabla\vec{u} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$
$$\frac{\partial v}{\partial t} + \vec{u}\cdot\nabla\vec{v} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$
$$\frac{\partial w}{\partial t} + \cdot\nabla\vec{w} + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w.$$

We further have the continuity equation that describes conservation of volume:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Now we scale each variable, using a representative value of horizontal velocity, vertical velocity, depth, horizontal length, pressure, coriolis parameter, and time. Each of those scales is written as a capital letter, and the *s have been dropped from the now non-dimensional variables. Scaled continuity:

$$\frac{U}{L}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{W}{H}\frac{\partial w}{\partial z} = 0.$$

Using this scaling, we can say that $W = UH/L$. Scaled momentum:

$$\frac{U}{T}\frac{\partial u}{\partial t} + \frac{U^2}{L}\vec{u}\cdot\nabla\vec{u} - UFv = -\frac{P}{\rho L}\frac{\partial p}{\partial x} + \nu\frac{U}{H^2}\nabla^2 u \tag{12.1}$$

$$\frac{U}{T}\frac{\partial v}{\partial t} + \frac{U^2}{L}\vec{u}\cdot\nabla\vec{v} + UFu = -\frac{P}{\rho L}\frac{\partial p}{\partial y} + \nu\frac{U}{H^2}\nabla^2 v \tag{12.2}$$

$$\frac{W}{T}\frac{\partial w}{\partial t}+\frac{UW}{L}\vec{u}\cdot\nabla\vec{w}+g=-\frac{P}{\rho H}\frac{\partial p}{\partial z}+\nu\frac{W}{H^2}\nabla^2 w. \quad (12.3)$$

Let us start by simplifying all the left hand terms in equation 1 (which are the same size in equation 2). Say that time scales advectively, so that T=L/U, then the first two terms scale the same, U^2/L. Letting U=0.1m/s and L=1000km=10^6m and $F = 10^{-4}s^{-1}$, typical of the Northern-hemisphere subtropical gyres, then the scale of the two left hand terms are:

$$[\frac{U^2}{L}] = 10^{-8}(m^2/s) \qquad [f_o U] = 10^{-5}(m^2/s).$$

Hence the Coriolis term is 3 orders of magnitude larger than the intertial/advective term. Compared to the Coriolis term the advective term is negligible. It is not conceivable that the terms on the left balance each other so something on the right must be what balances the Coriolis term. On the right we have the pressure gradient force and viscosity. If we let $H = 1000m$, $P = 10bar$, $\rho = 1000kg/m^3$, $\nu = 10^-3Pas$, then

$$[\frac{P}{\rho L}] = 10^{-3}(m^2/s) \qquad [\frac{\nu U}{H^2}] = 10^{-10}(m^2/s).$$

Again one term is much smaller, here the viscosity. It is even smaller than the advective term.

Dropping the advective and viscous terms (for the first-order balance) gives us with geostrophic balance in the horizontal:

$$f\hat{k}\times\vec{u} = -\frac{1}{\rho}\nabla_H p.$$

The third equation scales somewhat differently. From the continuity equation we know that W scales as UH/L. Again, we assume that time scales advectively so that W/T scales as UW/L, which is U^2H/L^2.

$$[\frac{U^2H}{L^2}] = 0.1^2 * 10^3/(10^6)^2 = 10^{-11}(m/s^2)$$

which is much smaller than $g \approx 10m/s^2$. Again, the viscosity will be small compared to the pressure gradient force, so the pressure term balances the gravity term and we are left with hydrostatic balance:

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

12.5 Nondimensionalizing three ways

Let's look at a simpler equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\kappa_h\frac{\partial^2 u}{\partial x^2}+\kappa_v\frac{\partial^2 u}{\partial z^2}=cu$$

The terms are, from left to right, the time rate of change of u, the horizontal and vertical advection of u, the horizontal and vertical diffusion of u and the linear drag. Replacing each variable with a characteristic unit (denoted here by capital letters) yields

$$\frac{1}{T}\frac{\partial u}{\partial t} + \frac{U}{L}u\frac{\partial u}{\partial x} + \frac{W}{H}w\frac{\partial u}{\partial z} + \frac{\kappa_h}{L^2}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa_v}{H^2}\frac{\partial^2 u}{\partial z^2} = cu$$

All variables are non-dimensional now though I have not changed the notation.

We also have the (2D) continuity equation, nondimensionally: $\frac{U}{L}\frac{\partial u}{\partial x} + \frac{W}{H}\frac{\partial w}{\partial z} = 0$. Using this scaling, we say that $W = UH/L$. Plugging this in:

$$\frac{1}{T}\frac{\partial u}{\partial t} + \frac{U}{L}u\frac{\partial u}{\partial x} + \frac{U}{L}w\frac{\partial u}{\partial z} + \frac{\kappa_h}{L^2}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa_v}{H^2}\frac{\partial^2 u}{\partial z^2} = cu$$

Notice that this causes both advective terms to scale in the same way. We now have to choose which time scale to use in nondimensionalizing. There are three options, the advective time scale, $t_A = L/U$, the diffusive time scales, $t_{Dh} = L^2/\kappa_h$, $t_{Dv} = H^2/\kappa_v$ and the frictional time scale, $t_f = 1/c$. This choice will depend on which processes you are interested in studying. A common choice for large scale circulation processes is the advective time scale, plugging this in and simplifying yields

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\kappa_h}{UL}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa_v L}{UH^2}\frac{\partial^2 u}{\partial z^2} = \frac{cL}{U}u$$

Now there are only non-dimensional parameters in front of the diffusive and frictional terms. If we had chosen to nondimensionalize using the horizontal diffusion time scale, we would get:

$$\frac{\partial u}{\partial t} + \frac{UL}{\kappa_h}(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}) + \frac{\partial^2 u}{\partial x^2} + \frac{\kappa_v L^2}{H^2\kappa_h}\frac{\partial^2 u}{\partial z^2} = \frac{cL^2}{\kappa_h}u$$

And nondimensionalizing using the frictional time scale:

$$\frac{\partial u}{\partial t} + \frac{U}{Lc}(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}) + \frac{\kappa_h}{L^2 c}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa_v}{H^2 c}\frac{\partial^2 u}{\partial z^2} = u$$

So, there are many ways to nondimensionalize, and it will come down to a choice that you make. Nothing changes about the system when you nondimensionalize, it is just a matter of making things easier to interpret for the *scale that you would be interested in.* Essentially, choosing the scales of interest gives you a nondimensional form that allows you to examine the limit of the equation at that scale and then you can choose to simplify the equation appropriately. For example, in the three cases above, deciding whether the nonlinear terms are important depends on different nondimensional numbers. In (1), they are important if $\frac{\kappa_h}{UL}$ or $\frac{\kappa_v L}{UH^2}$ are small. In (2), they are important if $\frac{UL}{\kappa_h}$ is large. In (3), they are important if $\frac{U}{Lc}$ is large.

The groupings of scale factors into nondimensional numbers leads to many of these nondimensional factors being used as a shorthand to describe the important terms in the system through whether they are large or small. I will now list a number of these terms which are commonly used.

12.5.1 Common Nondimensional Numbers

1. Rossby number $Ro = \frac{U}{fL}$ If it is small, the advective terms are small compared to the Coriolis term.
2. Ekman number $Ek = \sqrt{\frac{\nu}{fL^2}}$ Describes what fraction of the depth of the feature of interest is contained in the Ekman layer.
3. Reynolds number $Re = \frac{\rho LU}{\mu}$
4. Prandtl number $Pr = \frac{\nu}{\kappa}$ This is the ratio of momentum to heat diffusivity. (The Schmidt number is similar but for salinity.)
5. Peclet number $Pe = \frac{LU}{\kappa}$ This is the ratio of advective to diffusive timescales.
6. Burger number $B = \frac{NH}{fL}$ This is the ratio of importance of stratification to rotataion.
7. Richardson number $Ri = \frac{gH}{U^2}$ This is the ratio of potential to kinetic engergy.
8. Froude number $Fr = \frac{U^2}{gL}$ This is the ratio of inertial to gravitational force.

12.6 Fish Growth example

This example is based off the paper "Quantitative Analysis of Insect Outbreak Systems: The Spruce Budworm and Forest" by D. Ludwig, D. D. Jones and C.S. Holing in the Journal of Animal Ecology. Consider a fish population with fish density in a certain location given by B. The below equation describes the rate of change of fish density ($\frac{dB}{dt}$). The first term on the right is the growth rate, which is given by the logistic equation, chosen for simplicity; r_B is a constant growth constant and K_B is a carrying capacity, which depends on the environment. The second term on the right gives the loss rate; which approaches β as $B \to \infty$ and decays quadratically as $B \to 0$, α determines the scale of fish densities at which saturation begins to take place.

$$\frac{\mathrm{d}B}{\mathrm{d}t} = r_B B\left(1 - \frac{B}{K_B}\right) - \beta\frac{B^2}{\alpha^2 + B^2}.$$

To solve for stable points, set $\frac{dB}{dt} = 0$

$$r_B B\left(1-\frac{B}{K_B}\right)-\beta\frac{B^2}{\alpha^2+B^2}=0.$$

Clearly, B=0 is a stable solution, which means that the other solutions must satisfy:

$$r_B\left(1-\frac{B}{K_B}\right)-\beta\frac{B}{\alpha^2+B^2}=0.$$

Scaling B by the parameter α, $\mu = B/\alpha$ gives:

$$r_B\left(1-\frac{\alpha\mu}{K_B}\right)-\frac{\alpha\beta\mu}{\alpha^2(1+\mu^2)}=0.$$

Simplifying by multiplying through by α/B:

$$\frac{\alpha r_B}{\beta}\left(1-\frac{\alpha\mu}{K_B}\right)-\frac{\mu}{1+\mu^2}=0.$$

We can isolate two groups of dimensionless parameters and write our equation in terms of them.

$$R=\frac{\alpha r_B}{\beta},\quad Q=\frac{K_B}{\alpha}$$

$$R\left(1-\frac{\mu}{Q}\right)=\frac{\mu}{1+\mu^2}.$$

This form is more tractable and we can solve for stable points by finding the intersection of growth and loss rates, shown in Figure 12.1. Figure 12.1 shows the quadratic loss rate curve in light blue and linear growth rate lines for various values of R. There will always be one intersection and up to three (we are finding roots of a cubic function). The stability of these equilibrium intersection points can be assessed by the relative magnitudes of the growth and loss rates between equilibrium points. The stability of the R=0.45 equilibria points is illustrated in Figure 12.1. The highest and lowest equilibrium points are stable, whereas the central intersection is unstable. We can consider the significance of R and Q. From the linear growth function it is clear that R is its y-intercept and Q its x intercept. R is given by the ratio of α, the fish density scale, and r_B, the growth constant with the asymptotic loss rate (β), so that a large value of R implies better growth conditions. Q is given by the ratio of the carrying capacity and the fish density scale, so that the closer the system is to carrying capacity, the smaller Q will be and the lower the stable equilibrium point(s) will be.

It is also helpful to think about the units of all of these parameters, which can be found by looking at the first equation and ensuring that all terms have the same units.
B=fish/area; r_B= 1/time; K_B= fish/area; β= fish/area/time; α=fish/area
Notice that this confirms that R and Q are dimensionless.

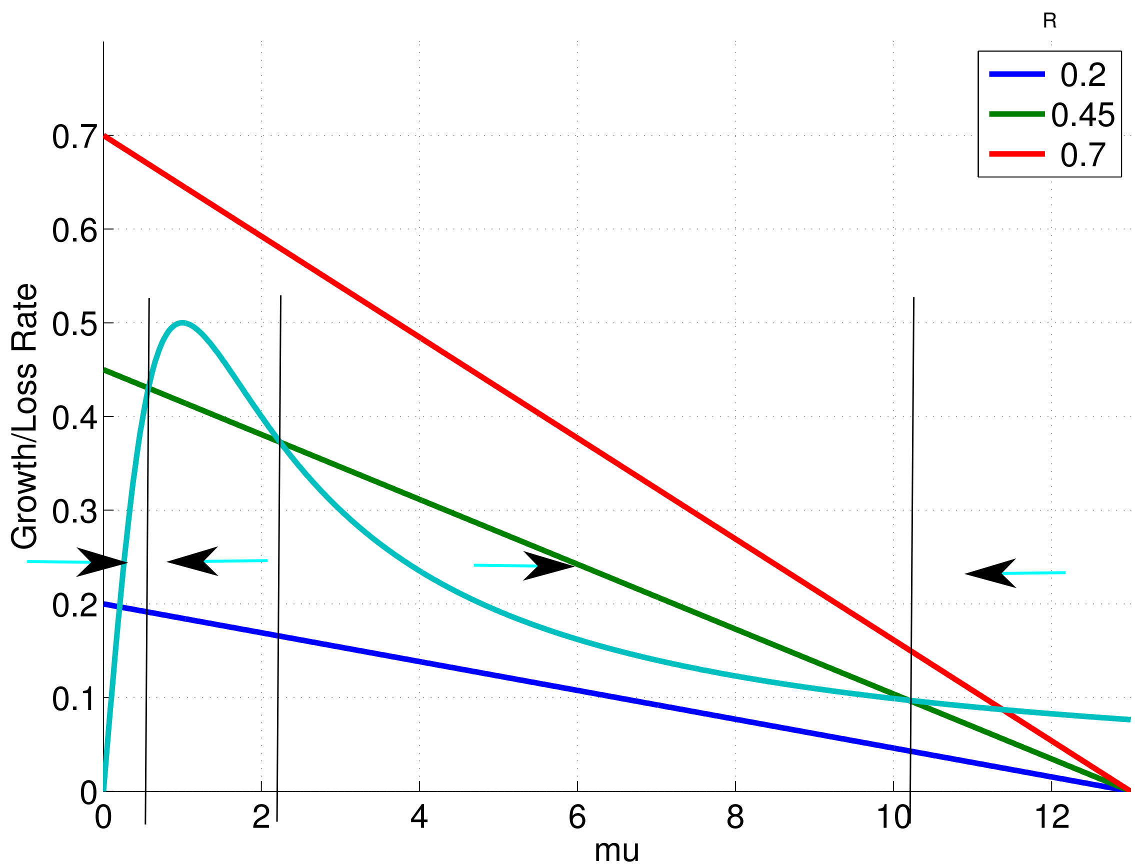

Figure 12.1: Fish growth rates (red, blue, green) and loss rate (light blue). Black lines indicate intersections for R=0.45 green growth rate line. Arrows show stability directions.