A Nonhydrostatic Mesoscale Ocean Model. Part I: Well-Posedness and Scaling

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ABSTRACT

The incompressibility and hydrostatic approximations that are traditionally used in large-scale oceanography to make the hydrodynamic equations more amenable to numerical integration result in the primitive equations. These are ill-posed in domains with open boundaries and hence not well-suited to mesoscale or regional modeling. Instead of using the hydrostatic approximation, the authors permit a greater deviation from hydrostatic balance than what exists in the ocean to obtain a system of equations that is well-posed with the specification of pointwise boundary conditions at open or solid boundaries. These equations, formulated with a free-surface boundary, model the mesoscale flow field accurately in all three-dimensions, even the vertical. It is essential to retain the vertical component of the Coriolis acceleration in the model since it is nonhydrostatic.

1. Introduction

The ocean is recognized to be a major component of the earth's climate system. Just the top few meters of the ocean's surface are capable of retaining and transporting more heat than the entire atmospheric layer. The ocean currents, which are largely responsible for the transport of heat and momentum, meander and form eddies in a highly irregular manner. A major challenge is to model this behavior over long periods of time. Eddy-resolving models developed for this purpose should be applicable to limited domains, since the resolution required by these models cannot be afforded for the entire world's oceans.

One fundamental difficulty with large-scale ocean modeling is the presence of waves with vastly different wave speeds. While we are interested in modeling the slower motions that constitute mesoscale features (features of 10–100 km in horizontal extent), it is the speed of the fastest waves that governs the time step of the explicit numerical integration scheme. Sound waves, the fastest waves in the ocean, propagate at speeds more than 10^3 times that of the fluid itself, and their presence requires the time step in the explicit numerical integration scheme to be 10^3 times smaller than what is required to resolve the motions of interest. Another fundamental problem in the three-dimensional modeling of large-scale oceanic flows is that the extremely small ratio of the vertical to horizontal length scales leads to a nearly perfect hydrostatic balance between the vertical pressure gradient and buoyancy forces. The slightest inaccuracy in computing the deviation from this balance results in a relatively large error in the vertical velocity obtained from integrating the vertical momentum equation. This makes the numerical solution of the equations for large-scale flow infeasible without the use of some approximations.

The approximations traditionally used to address these difficulties are by no means satisfactory. The incompressibility approximation filters out sound waves (or makes the sound speed infinite) and enables the use of a larger time step in the numerical procedure. It also alleviates the severe demand on accuracy in computing the divergence of the velocity. Computing the pressure is, however, at each time step relatively more expensive (especially in a nonuniform geometry) with the use of this approximation since it requires solving an elliptic equation or using some other iterative procedure in the absence of a time-dependent pressure equation. The fast surface gravity waves in the ocean are often eliminated by using the rigid-lid approximation in the model.

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The hydrostatic approximation, which assumes perfect balance between the vertical pressure gradient and fluid weight, is used to alleviate the unreasonable demand on accuracy in computing the vertical momentum. It eliminates acoustic waves in the vertical direction and can also be viewed as reducing the problem of computing the three-dimensional pressure field to a two-dimensional one (since once the pressure is known at one depth, it can be computed at any other depth by integrating the hydrostatic relation). A serious drawback of using the hydrostatic approximation in three-dimensional models is, however, that it renders the resulting (so called *primitive*) equations ill-posed in domains with open boundaries (Oliger and Sundstrom 1978; Browning et al. 1990). This can be seen from the fact that the sign of the eigenvalues, and hence the direction of the characteristics of the linearized system, is dependent on the wavenumber and is not known a priori at an open boundary. It is thus impossible to design pointwise boundary conditions at an open boundary for which the problem will be well-posed. A further deficiency in the hydrostatic equations is that since the vertical velocity is computed from the continuity equation, any error in computing the horizontal divergence of velocity is amplified by the inverse of the Rossby number in the vertical velocity. This error is significant in the small Rossby number flows of interest to us.

In spite of its ill-posedness, the *primitive* equation model continues to be used in domains with open boundaries. While effects of molecular viscosity are negligible in mesoscale and large-scale oceanographic flows, viscous terms are added to account for subgrid-scale motions and dissipate energy at smaller scales. The use of sufficiently large viscous terms makes the solution of the primitive equations computationally viable with open boundaries. The need to model the subgrid-scale motions, however, arises from the numerical discretization and not from the physics. Hence, we believe that the equations that describe the physics should not have to rely on a minimum size of viscous term for their well-posedness.

Although large-scale oceanographic flows are, to first approximation, hydrostatic and two-dimensional, the shallow-water and quasigeostrophic equations are unable to satisfactorily represent the energetic mesoscale flows associated with currents and eddies. It has become evident that even though the vertical velocities are much weaker than the horizontal velocities, it is necessary to model them accurately in order to explain mesoscale phenomena. Present day models are unable to achieve this.

The objective here is to develop and implement an accurate three-dimensional model for mesoscale flow that is well-posed in domains with open or solid boundaries. This paper is Part I of two parts and focuses on model development. Part II (Mahadevan et al. 1996) describes the numerical implementation and flow simulations.

In developing the model we use the approach of Browning et al. (1990) and refrain from making the usual hydrostatic approximation. Instead, we permit a greater deviation from hydrostatic balance than actually exists in the ocean to alleviate the stringent accuracy requirement in computing the vertical momentum. It thus becomes possible to compute the vertical velocity by integrating the vertical momentum equation. The modification to the equations does not have any significant effect on the large-scale motions of interest and the resulting equations are shown to maintain a desired solution accuracy. Most importantly, they are well-posed in domains with open or solid boundaries.

A scaling analysis of the governing equations reveals the existence of a fine balance between the Coriolis acceleration and nonhydrostatic pressure gradient in the vertical, analogous to the geostrophic balance in the horizontal. Hence, we find it essential to retain the commonly neglected component of the Coriolis acceleration in the vertical momentum equation.

In order to specify the correct boundary condition at the top surface and maintain the same desired relative accuracy in the vertical velocities as in the horizontal, we model the ocean with a free surface. The free-surface position is also of practical interest since satellite altimetry data of ocean surface elevation is readily available and can be used for assimilation into models. It turns out, for reasons explained in the following paper (Mahadevan et al. 1996), that the numerical integration of the free-surface model is much more efficient than that of the rigid-lid one.

The model is developed for mesoscale oceanic flows, for example the flow associated with the meandering Gulf Stream current that sheds eddies. The features of interest in such flows are typically 10–100 km in horizontal extent, occur over depths of order 1000 m and have characteristic horizontal velocities of 0.1–1 m s^{-1}. The earth's rotation has a dominant influence at such scales, and the Rossby number of these flows is of the order 10^{-2}–10^{-1}. The effects of molecular viscosity and molecular diffusion are 10^{-11}–10^{-10} times the inertial effects, and we have neglected them altogether. A subgrid-scale model that accounts for unresolved scales of motion and forcing terms that model heat fluxes, evaporation–precipitation and wind stresses omitted here for simplicity, may be added to the model without difficulty.

We believe that the model will enable us to capture the upwelling and downwelling in mesoscale flows that has eluded modelers. The transport of deep water to the surface has biological implications, and an understanding of this process would help piece together the overall picture of ocean circulation. The model is suited to studying the thermohaline structure of the ocean that is sensitive to the vertical circulation and to situations where varying topography may influence the vertical structure of the flow.

An outline of this paper is as follows. In sections 2 and 3 we present the governing equations and scale them based on the characteristics of mesoscale flow described above. In section 4 we introduce an increased compressibility and deviation from hydrostatic balance as in Browning et al. (1990). In section 5 we make the extension to incompressibility. In section 6 we reformulate the model with a free-surface boundary and show that we can achieve a desired accuracy with these equations in all three dimensions. Section 7 draws up the conclusions. The numerical implementation of the model and its application to simulating the flow in an ocean basin are described in Mahadevan et al. (1996).

2. Governing equations

Neglecting the effects of molecular viscosity, diffusion, and any sources or sinks of heat, salt, and momentum, the equations describing oceanographic flow can be written as

$$\frac{Ds}{Dt} = 0 \tag{2.1a}$$

$$\frac{DT}{Dt} = 0 \tag{2.1b}$$

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv + bw = 0 \tag{2.1c}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0 \tag{2.1d}$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{a} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g - bu = 0 \tag{2.1e}$$

$$\frac{Dp}{Dt} + c^2\rho\nabla\cdot\mathbf{V} = 0. \tag{2.1f}$$

$$\rho = \rho(s, T, p), \tag{2.1g}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

The coordinates x and y are eastward and northward distances along the surface of the globe. If ϕ denotes latitude, θ longitude, and a the mean radius of the earth, then $dx = a\cos\phi d\theta$ and $dy = ad\phi$. The variable z is the distance from the surface of the globe in a direction antiparallel to gravity. The components of velocity $\mathbf{V}$ are defined as $u \equiv dx/dt$, $v \equiv dy/dt$ in the eastward and northward directions, and $w \equiv dz/dt$ in the vertical direction. The Coriolis acceleration is resolved into a component normal to the earth's surface, $f \equiv 2\Omega\sin\phi$, and into a component tangential to the earth's surface, $b \equiv 2\Omega\cos\phi$, where ϕ is the latitude of the point where the equations are applied and Ω is the magnitude of the angular velocity of rotation of the earth.

In these equations s denotes salinity, T the potential temperature,[1] p the pressure, ρ the density, c the speed of sound, t time, and g acceleration due to gravity. Equation (2.1g) is an equation of state for density. The quadratic terms on the left-hand side of the momentum equations are due to the curvature of the earth (section 2.3, Holton 1992) and are relatively small.

3. Scaling the equations

In order to rewrite the governing equations in dimensionless form, we scale the variables (in a manner similar to Browning et al. 1990) as follows, such that the dimensionless (primed) quantities are $O(1)$.

$$\rho = R + R_0\rho_0'(z) + R_1\rho'(x, y, z) \tag{3.1a}$$

$$p = p_s - DGRg'z' + P_0p_0'(z) + P_1p'(x, y, z) \tag{3.1b}$$

$$x = Lx', \quad y = Ly', \quad z = Dz', \quad u = Uu'(x, y, z),$$
$$v = Uv'(x, y, z), \quad w = Ww'(x, y, z)$$
$$t = (L/U)t', \quad g = Gg', \quad a = Aa', \quad f = Ff'(y),$$
$$b = Fb'(y), \quad s = X + X_1s'(x, y, z),$$
$$T = Y + Y_1T'(x, y, z).$$

Here X, Y, R are the mean values of salinity, potential temperature, and density, and p_s is the surface pressure. The advective timescale L/U is chosen as the characteristic timescale of the motions. The values of the characteristic rotation rate $F = 10^{-4}\ \mathrm{s}^{-1}$, gravitational acceleration $G = 10\ \mathrm{m\ s}^{-2}$ and radius of earth $A = 10^7$ m.

The sum of the first three terms in (3.1b) is the hydrostatic pressure due to the mean density profile represented by the sum of the first two terms in (3.1a). It follows that

$$\frac{P_0}{D}\frac{\partial p_0'(z)}{\partial z'} + GR_0g'\rho_0'(z) = 0, \tag{3.2}$$

and hence

$$P_0 = DGR_0. \tag{3.3}$$

To estimate the size of the dimensionless groups of variables that arise from the scaling, we make use of the following characteristics of the flow:

1) It is approximately geostrophic, that is, the Rossby number $\epsilon \equiv U/FL \ll 1$ and the pressure gradient and Coriolis acceleration are in near balance. Hence, $P_1/RU^2 = FL/U = 1/\epsilon$. We consider the cases where $L = 10^4$ m, $U = 0.1\ \mathrm{m\ s}^{-1}$, and $L = 10^5$ m,

[1] The potential temperature T of a parcel of sea water is the temperature the parcel would have if it were displaced adiabatically to the sea surface; T is a conserved quantity and differs slightly from the in situ temperature due to the compressibility of sea water.

$U = 0.1 - 1\ \mathrm{m\ s^{-1}}$. For these cases the Rossby number ϵ lies in the range 0.1–0.01.

2) The ratio of the depth to length scale, $D/L \ll 1$. Hence the flow is nearly hydrostatic, that is, the vertical pressure gradient term is of the same order as the buoyancy term and $DGR_1/P_1 = 1$.

3) The Boussinesq approximation is justified and hence ρ/R can be taken to be 1 except where ρ is multiplied by g.

4) The speed of sound c is nearly constant in the fluid and hence can be replaced by the mean speed of sound C.

5) It occurs in the midlatitudes. Hence $\tan\phi = O(1)$.

These are the characteristics of a wide range of mesoscale flows. Features such as the Gulf Stream rings in the North Atlantic, eddies shed from the Loop Current in the Gulf of Mexico, and Mediterranean salt lenses, or "meddies", have typical diameters of 30–300 km, and typical surface speeds that range from $0.1\ \mathrm{m\ s^{-1}}$ to as much as $1\ \mathrm{m\ s^{-1}}$ in the Gulf Stream eddies.

By forming the equation for the vertical component of vorticity and nondimensionalizing it, we find that

$$\frac{W}{U}\frac{L}{D} = \epsilon. \tag{3.4}$$

We can now express Eq. (2.1) in terms of the dimensionless parameters in Table 1. Dropping the primes from the dimensionless quantities and neglecting terms less than or equal to $O(10^{-2})$, we rewrite the equations as follows:

$$\frac{Ds}{Dt} = 0 \tag{3.5a}$$

$$\frac{DT}{Dt} = 0 \tag{3.5b}$$

$$\frac{Du}{Dt} + \frac{1}{\epsilon}(p_x - fv + \epsilon\delta bw) = 0 \tag{3.5c}$$

$$\frac{Dv}{Dt} + \frac{1}{\epsilon}(p_y + fu) = 0 \tag{3.5d}$$

$$\frac{Dw}{Dt} + \frac{1}{\epsilon^2\delta^2}\left(p_z + \rho g - \delta bu - \lambda\epsilon\delta\frac{(u^2+v^2)}{a}\right) = 0 \tag{3.5e}$$

$$\frac{Dp}{Dt} + \frac{\epsilon}{\mathcal{M}^2}\left(u_x + v_y + \epsilon w_z + \epsilon\frac{\mathcal{M}^2}{\mathcal{F}_i^2}p_{0z}w - \epsilon\frac{\mathcal{M}^2}{\mathcal{F}^2}gw\right) = 0 \tag{3.5f}$$

$$\rho = \rho(s, T, p), \tag{3.5g}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \epsilon w\frac{\partial}{\partial z}.$$

TABLE 1. Dimensionless parameters in the inviscid equations for large-scale oceanographic flow. The range indicates the typical size of the parameters for the mesoscale flows discussed.

Parameter	Definition	Range
	$\lambda \equiv L/A$	10^{-2}–10^{-3}
Aspect ratio	$\delta \equiv D/L$	10^{-1}–10^{-2}
Rossby number	$\epsilon \equiv U/FL$	10^{-1}–10^{-2}
Mach number	$\mathcal{M} \equiv U/C$	10^{-3}–10^{-4}
Froude number	$\mathcal{F} \equiv U/(GD)^{1/2}$	10^{-2}–10^{-3}
Internal Froude number	$\mathcal{F}_i \equiv U/(R_0R^{-1}GD)^{1/2}$	10^{-1}–10^{-2}

The coefficients $1/\epsilon^2\delta^2$ and $\epsilon/\mathcal{M}^2$ in (3.5e) and (3.5f) lie in the range 10^4–10^8 and 10^5–10^7 for the relevant range of flow parameters in Table 1. In order to maintain two digits of accuracy in the equations, the groups of terms multiplied by these coefficients have to be computed to an accuracy of 0.5×10^{-2} times the inverse of the coefficient that multiplies them. This is an extremely stringent requirement on the accuracy of computation and is difficult to attain in practice. Since these coefficients serve to define a large sound speed, as shown in section 4a, their size also severely restricts the time step of the explicit numerical solution procedure. It is to overcome these difficulties that we need to resort to some approximations.

4. The quasi-compressible nonhydrostatic model

Instead of applying the hydrostatic and incompressibility approximations, we follow the approach of Browning et al. (1990) and reduce the sizes of the coefficients that multiply the groups of terms defining the hydrostatic balance and velocity divergence. This makes the fluid more compressible and permits a greater deviation from hydrostatic balance than actually exists, thus resulting in a decreased sound speed and internal gravity wave speed. Reducing the sound speed, an approach used in atmospheric flow simulations by Anderson et al. (1985) and Droegemeier and Wilhelmson (1987), permits a larger time step in the explicit numerical integration of the equations. The greater deviation from hydrostatic balance and the greater compressibility alleviates the unreasonably stringent accuracy requirement in computing the hydrostatic balance and velocity divergence.

To make this approximation we replace the large coefficients $1/\epsilon^2\delta^2$ and $\epsilon/\mathcal{M}^2$ in (3.5e) and (3.5f) by α_1, α_2, where

$$\alpha_1 \ll 1/\epsilon^2\delta^2, \quad \alpha_2 \ll \epsilon/\mathcal{M}^2. \tag{4.1}$$

Browning et al. (1990) showed from scaling that the error made in using the modified coefficients α_1 and α_2 is no greater than $\max[\alpha_1^{-1}, (\epsilon\alpha_2)^{-1}]$. Thus, if we choose

$$\alpha_1 = 10^2, \quad \alpha_2 = 10^2\epsilon^{-1} \tag{4.2}$$

we make an error no larger than $O(10^{-2})$. In the approximate system that uses α_1, α_2 the sound speed is several orders of magnitude less than the true sound speed. Hence we can use a larger time step in the explicit numerical solution of the approximate system, which is written as

$$\frac{Ds}{Dt} = 0 \qquad (4.3a)$$

$$\frac{DT}{Dt} = 0 \qquad (4.3b)$$

$$\frac{Du}{Dt} + \frac{1}{\epsilon}(p_x - fv + \epsilon\delta bw) = 0 \qquad (4.3c)$$

$$\frac{Dv}{Dt} + \frac{1}{\epsilon}(p_y + fu) = 0 \qquad (4.3d)$$

$$\frac{Dw}{Dt} + \alpha_1(p_z + \rho g - \delta bu) = 0 \qquad (4.3e)$$

$$\frac{Dp}{Dt} + \alpha_2\left(u_x + v_y + \epsilon w_z - \epsilon\frac{\mathcal{M}^2}{\mathcal{G}^2} gw\right) = 0 \qquad (4.3f)$$

$$\rho = \rho(s, T, p). \qquad (4.3g)$$

Terms that make a contribution less than or equal to $O(10^{-2})$ for the values of α_1, α_2 suggested in (4.2) have been neglected. While Browning et al. (1990) use a conservation of mass equation for density, we prefer to model the physically relevant salinity and temperature fields in the ocean, and use and equation of state for density. This also avoids the problems that arise because the conservation of mass and pressure equations reduce to the same equation in the incompressibility limit.[2]

We should mention that the approximation used here is opposite in approach to the hydrostatic and incompressibility approximations that take the limit as α_1 and α_2 approach infinity, respectively. The accuracy of the hydrostatic and incompressibility approximations and some suggested improvements are discussed in the appendix. Another noteworthy improvement to the hydrostatic approximation is the "quasihydrostatic" approximation proposed by Orlanski (1981).

a. Sound speed

Equations (4.3) are a quasilinear hyperbolic set of equations that can be written in the form

$$\frac{\partial \mathbf{q}}{\partial t} + A_1(\mathbf{q})\frac{\partial \mathbf{q}}{\partial x} + A_2(\mathbf{q})\frac{\partial \mathbf{q}}{\partial y} + A_3(\mathbf{q})\frac{\partial \mathbf{q}}{\partial z} + \mathbf{F}(\mathbf{q}) = 0, \qquad (4.4)$$

where

$$\mathbf{q} = (s, T, u, v, w, p)^{\mathrm{T}},$$

$A_1(\mathbf{q}), A_2(\mathbf{q}), A_3(\mathbf{q})$ are coefficient matrices and $\mathbf{F}(\mathbf{q})$ is the forcing vector. The eigenvalues of A_1, A_2, A_3 are

$$u,\ u,\ u,\ u,\ u - \sqrt{\alpha_2\epsilon^{-1}},\quad u + \sqrt{\alpha_2\epsilon^{-1}},$$

$$v,\ v,\ v,\ v,\ v - \sqrt{\alpha_2\epsilon^{-1}},\quad v + \sqrt{\alpha_2\epsilon^{-1}},$$

and

$$\epsilon w,\ \epsilon w,\ \epsilon w,\ \epsilon w,\ \epsilon w - \sqrt{\epsilon\alpha_1\alpha_2},\quad \epsilon w + \sqrt{\epsilon\alpha_1\alpha_2},$$

respectively. Since the eigenvalues are real, the system is hyperbolic. The largest eigenvalue is the speed of the fastest wave and governs the time step in the numerical integration of the equations when an explicit method is used. Here $\sqrt{\alpha_2\epsilon^{-1}}$ is the sound speed in the horizontal directions nondimensionalized by U, while $\sqrt{\epsilon\alpha_1\alpha_2}$ is the sound speed in the vertical direction nondimensionalized by $\epsilon^{-1}W$. Table 2 illustrates the changes made to the equations and correspondingly to the sound speed by the approximations.

Decreasing the coefficient in the vertical momentum equation in addition to the coefficient in the pressure equation results in the modified sound speed being less in the vertical direction than in the horizontal.

b. Well-posedness

We can symmetrize the equations (4.3) by making the substitutions

$$\tilde{u} = \sqrt{\epsilon\alpha_2}\,u,\quad \tilde{v} = \sqrt{\epsilon\alpha_2}\,v,\quad \tilde{w} = \sqrt{\frac{\epsilon\alpha_2}{\alpha_1}}\,w.$$

The system then becomes

$$\frac{\partial \tilde{\mathbf{q}}}{\partial t} + \tilde{A}_1(\tilde{\mathbf{q}})\frac{\partial \tilde{\mathbf{q}}}{\partial x} + \tilde{A}_2(\tilde{\mathbf{q}})\frac{\partial \tilde{\mathbf{q}}}{\partial y} + \tilde{A}_3(\tilde{\mathbf{q}})\frac{\partial \tilde{\mathbf{q}}}{\partial z} + \tilde{\mathbf{F}}(\tilde{\mathbf{q}}) = 0, \qquad (4.5)$$

where

$$\tilde{\mathbf{q}} = (s, T, \tilde{u}, \tilde{v}, \tilde{w}, p)^{\mathrm{T}}.$$

[2] If the speed of sound is treated as a constant, the equation for pressure and the conservation of mass equation are linearly dependent for an adiabatic flow. In the Browning et al. (1990) model, the divergence of velocity is substituted into the conservation of mass equation from the pressure equation to remove this dependency; in the process a "potential density" ρ_p is defined and density ρ is replaced by $\rho_p \equiv \rho - 10^{-2}p$ where p is the pressure. While this process eliminates the dependency of the equations, the introduction of ρ_p in lieu of ρ in the vertical momentum equation introduces an error that is amplified by the large coefficient that precedes it. This error could be eliminated by substituting $\rho_p + 10^{-2}p$ for ρ in the vertical momentum equation.

TABLE 2. Values of coefficients multiplying the hydrostatic balance and velocity divergence in the vertical momentum and pressure equations. The corresponding sound speeds in the vertical and horizontal directions are indicated.

	Value of coefficient in		Sound speed (m s^{-1})	
	Vertical momentum equation	Pressure equation	Horizontal	Vertical
Original equations	$1/\epsilon^2\delta^2$	$\epsilon/\mathcal{M}^2$	C	C
Primitive equations	infinity	infinity	infinity	infinity
Approximate equations	α_1	α_2	$\sqrt{\epsilon^{-1}\alpha_2}U$	$\sqrt{\epsilon\alpha_1\alpha_2}\epsilon^{-1}W$
Suggested (for two digit accuracy)	100	$\epsilon^{-1}100$	$\epsilon^{-1}100U$	$\epsilon^{-1}100W = 100\delta U$
when $U = 1$ m s^{-1}, $L = 10^5$ m	100	1000	100	1
when $U = 0.1$ m s^{-1}, $L = 10^5$ m	100	10 000	100	0.1
when $U = 0.1$ m s^{-1}, $L = 10^4$ m	100	1000	10	1

Now $\tilde{A}_1(\tilde{\mathbf{q}})$, $\tilde{A}_2(\tilde{\mathbf{q}})$, $\tilde{A}_3(\tilde{\mathbf{q}})$ are symmetric coefficient matrices and $\tilde{\mathbf{F}}(\tilde{\mathbf{q}})$ represents the modified forcing vector.

Since Eqs. (4.3) may be reduced to symmetric hyperbolic system, the differential operator is semibounded (Kreiss and Lorenz 1989). The initial–boundary value problem is well-posed if the incoming characteristic variables are specified (Oliger and Sundstrom 1978).

c. Boundary conditions

To see what boundary conditions need to be specified, we rewrite the equations in characteristic form. To do this we must diagonalize the coefficient matrices in (4.4). Unfortunately A_1, A_2, A_3 cannot all be diagonalized simultaneously, and the analysis must be done one direction at a time. The direction for the characteristic is usually chosen as the direction normal to the boundary, with the assumption that information propagates normal to an open boundary. Here we choose x as the characteristic direction and diagonalize A_1 to get the characteristic equations.

We neglect the terms $A_2(\mathbf{q})\partial\mathbf{q}/\partial y + A_3(\mathbf{q})\partial\mathbf{q}/\partial z$ from Eq. (4.4), or alternatively move them over to the right-hand side (which we have not done to retain simplicity) and rewrite the equation

$$\frac{\partial \mathbf{q}}{\partial t} + A_1(\mathbf{q})\frac{\partial \mathbf{q}}{\partial x} + \mathbf{F}(\mathbf{q}) = 0 \tag{4.6}$$

as

$$\frac{\partial \mathbf{q}}{\partial t} + X\Lambda X^{-1}\frac{\partial \mathbf{q}}{\partial x} + \mathbf{F}(\mathbf{q}) = 0, \tag{4.7}$$

where the columns of matrix X are eigenvectors of A_1, and Λ is the diagonal matrix of eigenvalues of A_1. The characteristic equations are given by

$$X^{-1}\frac{\partial \mathbf{q}}{\partial t} + \Lambda X^{-1}\frac{\partial \mathbf{q}}{\partial x} = -X^{-1}\mathbf{F}(\mathbf{q}). \tag{4.8}$$

This is the system of equations

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0 \tag{4.9a}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \tag{4.9b}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = -\frac{fu}{\epsilon} \tag{4.9c}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} = -\alpha_1(\rho g - \delta b u) \tag{4.9d}$$

$$\frac{\partial}{\partial t}\left(u + \frac{p}{\sqrt{\alpha_2\epsilon}}\right) + \left(u + \sqrt{\frac{\alpha_2}{\epsilon}}\right)\frac{\partial}{\partial x}\left(u + \frac{p}{\sqrt{\alpha_2\epsilon}}\right) = \frac{fv}{\epsilon} - \delta b w. \tag{4.9e}$$

$$\frac{\partial}{\partial t}\left(u - \frac{p}{\sqrt{\alpha_2\epsilon}}\right) + \left(u - \sqrt{\frac{\alpha_2}{\epsilon}}\right)\frac{\partial}{\partial x}\left(u - \frac{p}{\sqrt{\alpha_2\epsilon}}\right) = -\frac{fv}{\epsilon} + \delta b w. \tag{4.9f}$$

The quantity $\sqrt{\alpha_2/\epsilon}$, which is the dimensionless sound speed in the x direction, is much larger than u. At a solid boundary $x = x_b$, the normal velocity u is zero. The only incoming characteristic is (4.9f), and hence only one boundary condition, $u = 0$, need be specified. At an inflow boundary $x = x_b$, there are five incoming characteristics, (4.9a)–(4.9e), and five boundary conditions need to be specified. Four of them are s, T, v, w and the fifth is either u or p, or a combination of u and p. At an outflow boundary, there is one incoming characteristic, that is, (4.9f), and one boundary condition needs to be specified. This could be either the normal velocity u, the pressure p, or a combination of u and p. With these boundary conditions, the problem is well-posed.

d. Difficulties in the numerical solution

If we consider a flow characterized by $U = 1$ m s^{-1}, $L = 10^5$ m and $\epsilon = 0.1$, the reduced horizontal sound speed in the approximate equations that maintain 2 digits of accuracy is 100 m s^{-1} as seen from Table 2. Though this is less than one-tenth the actual sound speed in the ocean, it is 10^2 times the characteristic horizontal velocity, and the time step required in the explicit numerical procedure is 10^2 times smaller than the time step required to resolve the advective motions. While decreasing α_2 further would further decrease the sound speed, we cannot do this as it would lead to a loss of accuracy in the solution.

We attempted to integrate the approximate equations using an explicit time-splitting method that uses smaller time steps for the fast part of the equations associated with the large eigenvalues, and larger time steps for the slower parts (LeVeque and Oliger 1983). Such methods have been used for atmospheric flows by Klemp and Wilhelmson (1978). The ratio of the large to small time steps is equal to the ratio of the modified sound speed to advective speed, or the ratio of largest to small eigenvalues and is equal to 100 in this case. The method fails for this large a ratio. Numerical experiments by Skamarock and Klemp (1992) reveal that time split schemes work only when the ratio of the fast to slow motions is on the order of 10 or less. We also had problems with maintaining accuracy in the integration of the pressure equation where $\alpha_2 = 1000$ and the divergence of the velocity needed to be computed to an accuracy of 0.5×10^{-5} to maintain the two digits of accuracy that we are striving for.

5. The incompressible nonhydrostatic model

We now reconsider the approximation to the pressure equation on account of the problems faced in the numerical solution of the quasi-compressible equations. We reverse the trend of the approximation in the pressure equation, which was to decrease the size of the large coefficient $\epsilon/\mathcal{M}^2$ to α_2. Instead we take the limit as $\mathcal{M} \to 0$. In the vertical momentum equation we replace $1/\epsilon^2\delta^2$ by a smaller coefficient α_1 as before. The equation for pressure is thus modified into

$$u_x + v_y + \epsilon w_z = 0, \tag{5.1a}$$

a constraint arising from the incompressibility, and the equation of state

$$\rho = \rho(s, T) \tag{5.1b}$$

is now for the potential density,[3] which is independent of pressure.

[3] The potential density ρ of a parcel of sea water is the density the parcel would have if it were displaced adiabatically to the sea surface.

The numerical solution of Eqs. (5.1) along with (4.3a–e) requires an iterative procedure since they are no longer a hyperbolic system and the time-dependent pressure equation has been replaced by a constraint on the velocity. A common method is to construct an elliptic equation for pressure whose solution, when used in the momentum equations, results in a divergence-free velocity field. The additional work required in solving an elliptic equation at each time step is fortunately offset by the fact that the time step of the explicit numerical integration is no longer governed by the sound speed since the equations do not admit acoustic waves.

Well-posedness and boundary conditions for the incompressible equations

The incompressible equations (5.1) along with (4.3a–e) arise as the singular limiting equations for the compressible system as the Mach number tends to zero. The solution approximates the solution of the compressible equations provided the Mach number is small as is the case in the ocean. This is shown formally by the use of asymptotic expansions by Majda (1984). A restriction on the initial condition for the incompressible equations is that the flow field be divergence free. The incompressible system is thus well-posed with the specification of the boundary conditions described for the compressible case.

According to Oliger and Sundstrom (1978), rigorous results on the necessary form of the boundary conditions for the well-posedness of systems like (5.1), (4.3a–e) are obtainable. The problem is well-posed if the velocity vector, salinity, and temperature are specified at the inflow and the normal velocity is specified at the outflow. At solid boundaries, the condition is that the normal velocity is zero. Nothing further need be specified. The compressible system is well-posed when either the pressure, or the normal velocity is specified at the outflow. Numerical experiments suggest that this is also the case for the incompressible equations.

6. The free-surface model

It is common practice to approximate the free surface by a rigid lid since free-surface variations are small compared to the depth of the ocean. The advantage in doing this is that the position of the top boundary is fixed and that one need not worry about the time step of the numerical integration being restricted by surface gravity waves whose speeds are about 10^2–10^3 times that of the fluid. However, there has recently been considerable interest in free-surface models (Dukowicz and Smith 1994; Killworth et al. 1991; Blumberg and Herring 1987) because data of ocean surface elevation available from satellite altimetry could be used for assimilation and model validation. It has also become a matter of contention as to whether rigid-lid models are able to model the physics and energetics of mesoscale flows.

Our motivation to model the free surface stems from wanting to be accurate in the specification of boundary conditions at the top boundary. The nonhydrostatic equations model the dimensionless vertical velocity to two (or more) digits accuracy[4] provided the boundary conditions are accurate. Obviously we could not claim this accuracy if we were to use the rigid-lid approximation and introduce an $O(1)$ error in the vertical velocity w by setting $w = 0$ at the top surface.

It turns out that the free-surface formulation used here lends a very substantial numerical advantage to the solution of the three-dimensional (3D) elliptic equation for pressure that we solve at each time step in the solution of the nonhydrostatic equations. We decouple the pressure into a hydrostatic and nonhydrostatic component. The hydrostatic component is obtained by solving a two-dimensional (2D) boundary value problem for the free surface using a semi-implicit formulation. The small aspect ratio of the ocean gives the nonhydrostatic pressure field the quality that its second derivative varies much more over the horizontal extent of the domain than over the vertical. Hence, with the exaggerated deviation from hydrostatic balance, the 3D elliptic equation for the nonhydrostatic component of the pressure is much easier to solve than the 3D elliptic equation for the total pressure in the rigid-lid model. Further, we can use a Dirichlet boundary condition at the free surface for the nonhydrostatic pressure as opposed to having to use a Neumman boundary condition for the total pressure in the rigid-lid case. The numerical solution of the nonhydrostatic model with free surface that is described in Part II of this paper, requires only about one-tenth the computational effort required for the same model with a rigid lid, and is just a few times more expensive than the ill-posed hydrostatic model. The free-surface formulation hence dispels our reservations in using the nonhydrostatic equations.

a. Governing equations for free-surface flow

In the free-surface model, the position of the free-surface $h(x, y, t)$, is a variable in the problem. Integrating the equation of continuity from the lower boundary $z = -d$ to the free surface $z = h$, and using the kinematic boundary conditions we get

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_{-d}^{h} u dz\right) + \frac{\partial}{\partial y}\left(\int_{-d}^{h} v dz\right) = 0. \quad (6.1)$$

This equation for h (expressed in dimensional form) is an equation of continuity for a column or vertical line of incompressible fluid extending from the ocean bottom to the free surface.

In the free-surface formulation, we can compute the hydrostatic pressure at any point by integrating the weight of the fluid downward from the free surface once the free-surface position is known. We thus decompose the pressure p into a hydrostatic component p_h and a nonhydrostatic component q (Casulli 1995) as follows:

$$p = p_h + q. \quad (6.2)$$

The dimensional forms of horizontal momentum equations (2.1c) and (2.1d), written while neglecting some of the small quadratic curvature terms, are then

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial}{\partial x}(p_h + q) - fv + bw = 0 \quad (6.3a)$$

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial}{\partial y}(p_h + q) + fu = 0, \quad (6.3b)$$

where

$$f \equiv 2\Omega \sin\phi, \quad b \equiv 2\Omega \cos\phi.$$

Since the hydrostatic pressure p_h satisfies the relationship

$$\frac{1}{\rho}\frac{\partial p_h}{\partial z} = -g \quad (6.5)$$

by definition, the vertical momentum equation (2.1e) reduces to

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{a} + \frac{1}{\rho}\frac{\partial q}{\partial z} - bu = 0. \quad (6.6)$$

b. Scaling

We now nondimensionalize the momentum equations (6.3a), (6.3b), and (6.6) using

$$x = Lx', \quad y = Ly', \quad z = Dz', \quad t = (L/U)t',$$
$$u = Uu', \quad v = Uv', \quad w = Ww'$$
$$q = Qq', \quad p_{hx} = (P/L)p'_{hx'}, \quad p_{hy} = (P/L)p'_{hy'},$$
$$a = Aa', \quad f = Ff', \quad b = Fb'$$

so that the dimensionless (primed) quantities and their derivatives are $O(1)$. The quantities L, D, U, W, F, A are the same as in section 3 and represent the characteristic length, depth, horizontal velocity, vertical velocity, Coriolis parameter value, and radius of the earth; P is the characteristic variation in hydrostatic pressure over the horizontal length scale L and Q is the characteristic value of the nonhydrostatic pressure q that varies about 0. The potential density, expressed as the mean density R plus a deviation about the mean, is written as

[4] We measure accuracy in terms of the dimensionless variables because the velocities in the vertical direction are several orders smaller than in the horizontal ($W = \epsilon\delta U$).

$$\rho = R + R_1\rho'(x, y, z, t).$$

We need to determine the characteristic size of the nonhydrostatic pressure Q in terms of the other variables. On making the Boussinesq approximation and dropping the primes from the dimensionless variables, we write the dimensionless momentum equations as

$$\frac{Du}{Dt} + \frac{1}{\epsilon}(p_{hx} + \tilde{\delta}q_x - fv + \epsilon\delta bw) = 0 \quad (6.7a)$$

$$\frac{Dv}{Dt} + \frac{1}{\epsilon}(p_{hy} + \tilde{\delta}q_y + fu) = 0 \quad (6.7b)$$

$$\frac{Dw}{Dt} + \frac{1}{\epsilon^2\delta}\left(\frac{\tilde{\delta}}{\delta}q_z - bu - \epsilon\lambda\frac{(u^2 + v^2)}{a}\right) = 0, \quad (6.7c)$$

where

$$\tilde{\delta} \equiv Q/P,$$

$\delta \equiv D/L$ is the aspect ratio, $\lambda \equiv L/A$, and ϵ is the Rossby number as before. From the arguments presented in section 3, we know that

$$\frac{P}{RU^2} = \frac{FL}{U} \equiv \frac{1}{\epsilon} \quad \text{and} \quad \frac{W}{U} = \epsilon\delta.$$

Clearly in (6.7c), the vertical Coriolis acceleration term that is $O(1/\epsilon^2\delta)$ must be balanced by the nonhydrostatic pressure gradient, which must also be of the same order in magnitude. Hence

$$\tilde{\delta} = \delta. \quad (6.8)$$

Thus, we see that the vertical Coriolis acceleration and nonhydrostatic pressure gradient are the dominant terms in the vertical momentum equation. They are in balance and their difference results in vertical acceleration. Omitting either of these terms would leave a large unbalanced term; therefore, the Coriolis acceleration due to the tangential component of the earth's rotation cannot be neglected in the nonhydrostatic equations. The ratio of the nonhydrostatic to hydrostatic pressure gradients is of the order of the aspect ratio δ.

c. The quasi-nonhydrostatic approximation

Already we can see an advantage from decomposing the pressure into its hydrostatic and nonhydrostatic components in the free-surface model. The $O(1)$ acceleration term in the vertical momentum equation is equal to the difference between the nonhydrostatic pressure gradient and Coriolis terms, each of which are $O(1/\epsilon^2\delta)$. In the rigid-lid equations (3.5), the $O(1)$ vertical acceleration was the difference between the $O(1/\epsilon^2\delta^2)$ pressure gradient and buoyancy terms plus the effect of the $O(1/\epsilon^2\delta)$ Coriolis term. By subtracting out the $O(1/\epsilon^2\delta^2)$ pressure gradient and buoyancy effects that cancel each other out, we are left with $O(1/\epsilon^2\delta)$ terms and gain in the accuracy of computation.

As in the rigid-lid model of section 5, we would like to reduce the size of the coefficient $1/\epsilon^2\delta$ in the vertical momentum equation (6.7c) to further alleviate the demand on accuracy. We therefore replace $1/\epsilon^2\delta$ by β, where

$$\beta \ll 1/\epsilon^2\delta, \quad (6.9)$$

and making use of (6.8) rewrite the vertical momentum equation as

$$\frac{Dw}{Dt} + \beta\left(q_z - bu - \epsilon\lambda\frac{(u^2 + v^2)}{a}\right) = 0. \quad (6.10)$$

In section 4 it was stated, in accordance with Browning et al. (1990), that the maximum error in modifying the coefficient $1/\epsilon^2\delta^2$ that multiplies the difference between the pressure gradient, buoyancy, and Coriolis effect to α_1 is $O(\alpha_1^{-1})$. We had therefore suggested using $\alpha_1 = 100$ to maintain two digits of accuracy. Here the maximum error involved in modifying the coefficient $1/\epsilon^2\delta$ to β is $O(\beta^{-1})$. Hence we choose $\beta = 100$ to maintain two digits of solution accuracy in the dimensionless variables.

Any error in the computation of $(p_z + \rho g - \delta bu)$ in (4.3e) is amplified by α_1, which is taken to be 100. In (6.10) we have reduced the error by subtracting out the buoyancy term and pressure gradient that balances it.

d. Accuracy

We now present a sketch of the scaling analysis that shows that the maximum error in using the modified coefficient β is $O(\beta^{-1})$. This analysis is in direct analogy to the scaling analysis of Browning et al. (1990) that determined the minimum size of α_1, α_2 in section 4.

The dimensionless form of the momentum equations is given by (6.7), where $\tilde{\delta} = \delta$. Modifying the coefficient in (6.7c) to β results in the approximate equations

$$\frac{D\bar{u}}{Dt} + \frac{1}{\epsilon}(\bar{p}_{hx} + \delta\bar{q}_x - f\bar{v} + \epsilon\delta b\bar{w}) = 0 \quad (6.11a)$$

$$\frac{D\bar{v}}{Dt} + \frac{1}{\epsilon}(\bar{p}_{hy} + \delta\bar{q}_y + f\bar{u}) = 0 \quad (6.11b)$$

$$\frac{D\bar{w}}{Dt} + \beta\left(\bar{q}_z - b\bar{u} - \lambda\epsilon\frac{(\bar{u}^2 + \bar{v}^2)}{a}\right) = 0. \quad (6.11c)$$

The overbars on the variables denote that they are solutions to the approximate equations. Rewriting the exact vertical momentum equation (6.7c) as

$$\frac{Dw}{Dt} + \beta\left(q_z - bu - \epsilon\lambda\frac{(u^2 + v^2)}{a}\right) = F, \quad (6.11d)$$

where

$$F \equiv \left(\beta - \frac{1}{\epsilon^2\delta}\right)\left[q_z - bu - \epsilon\lambda\left(\frac{u^2+v^2}{a}\right)\right]$$

and then subtracting the approximate equations (6.11a), (6.11b), (6.11c) from the exact equations (6.7a), (6.7b), (6.11d) gives

$$\frac{Du_0}{Dt} + \frac{1}{\epsilon}(p_{h0x} + \delta q_{0x} - fv_0 + \epsilon\delta b w_0) = 0 \quad (6.12a)$$

$$\frac{Dv_0}{Dt} + \frac{1}{\epsilon}(p_{0hy} + \delta q_{0y} + fu_0) = 0 \quad (6.12b)$$

$$\frac{Dw_0}{Dt} + \beta\left(q_{0z} - bu_0 - \lambda\epsilon\frac{(u_0^2+v_0^2)}{a}\right) = F, \quad (6.12c)$$

where the variables

$$u_0 = u - \bar{u}, \quad v_0 = v - \bar{v}, \quad w_0 = w - \bar{w},$$
$$p_{0h} = p_h - \bar{p}_h, \quad q_0 = q - \bar{q}$$

denote the solution error and have homogeneous initial and boundary conditions. Now

$$F = (1 - \epsilon^2\delta\beta)\frac{Dw}{Dt}$$
$$\approx \frac{Dw}{Dt} = O(1) \quad (6.13)$$

since $\beta \ll 1/\epsilon^2\delta$. Therefore, neglecting terms that are $O(\epsilon)$, $O(\delta)$ or less, we get

$$p_{h0x} - fv_0 = 0 \quad (6.14a)$$
$$p_{h0y} + fu_0 = 0 \quad (6.14b)$$
$$q_{0z} - bu_0 = \beta^{-1}F \quad (6.14c)$$

from the momentum equations, and

$$w_{0z} = 0 \quad (6.14d)$$

from the continuity equation treated in a similar manner. Thus,

$$p_{h0x} = p_{h0y} = u_0 = v_0 = 0. \quad (6.15)$$

The solution to the error equations (6.12) is $O(\beta^{-1})$. Hence the maximum error in using the approximate equations (6.11) is also $O(\beta^{-1})$.

By forming the equation for the kinetic energy (KE) $(u^2 + v^2 + w^2)\rho/2$ (expressed here in dimensional form) for the approximate equations, one observes a contribution to the KE $-buw(1 - C)$ from the Coriolis term. Here C is the ratio between the modified coefficient β in the model and the dimensionless parameter $1/\epsilon^2\delta$ in the original equations. This results from the asymmetry introduced by the approximation. In the exact equations $C = 1$ and the Coriolis terms do no work, as expected. In the approximate equations $C \ll 1$ $= O(10^{-2} - 10^{-6})$. However, if we compute the time rate of change of the normalized KE as introduced by this term, the normalized rate is always less than $O(10^{-6})$ for the parameters of this paper; this is clearly negligible. Since KE $\sim u^2$, the error in the KE behaves like the square of the error in the velocity variables and is consistent with what is permitted in the model.

Since the term bw in the x momentum equation is formally negligible at the level of accuracy of the analysis, it is also perfectly appropriate to modify this term by multiplying it by C when we approximate the z momentum equation. Then the Coriolis terms balance as desired and indeed do zero work.

e. Hydrostatic pressure and free-surface elevation

The hydrostatic pressure p_h can be computed by integrating the weight of fluid downward from the free surface. Reverting once more to the dimensional form of the variables and indicating the dimensionless quantities by primes, we express the hydrostatic pressure $p_h(x, y, z, t)$ as

$$p_h = \int_z^h \rho g dz = \int_z^h (R + R_1\rho')g dz$$
$$= Rg(h - z) + g\int_z^h R_1\rho' dz. \quad (6.16)$$

We now define H as the characteristic free-surface variation over a horizontal distance L, so that the gradient of h is nondimensionalized as

$$h_x = (H/L)h'_{x'}, \quad h_y = (H/L)h'_{y'},$$

where $h'_{x'}$, $h'_{y'}$ are $O(1)$. Differentiating (6.16) with respect to x and y, substituting into the horizontal momentum equations, and nondimensionalizing we get

$$\frac{Du'}{Dt'} + \frac{P}{RU^2}\left(\frac{RGH}{P}g'h'_{x'} + r'_x + \tilde{\delta}q'_x\right) - \frac{1}{\epsilon}f'v' + \delta b'w' = 0 \quad (6.17a)$$

$$\frac{Dv'}{Dt'} + \frac{P}{RU^2}\left(\frac{RGH}{P}g'\mathrm{h}'_{y'} + r'_y + \tilde{\delta}q'_{y'}\right) + \frac{1}{\epsilon}f'u' = 0, \quad (6.17b)$$

where

$$r'_{x'} \equiv \frac{R_1GD}{P}g'\frac{\partial}{\partial x'}\int_{z'}^{(H/D)h'}\rho' dz'$$
$$= \frac{\epsilon}{\mathcal{F}_i^2}g'\frac{\partial}{\partial x'}\int_{z'}^{(H/D)h'}\rho' dz' \quad (6.18a)$$

$$r'_{y'} \equiv \frac{R_1 GD}{P} g' \frac{\partial}{\partial y'} \int_{z'}^{(H/D)h'} \rho' dz'$$

$$= \frac{\epsilon}{\mathcal{F}_i^2} g' \frac{\partial}{\partial y'} \int_{z'}^{(H/D)h'} \rho' dz' \quad (6.18b)$$

are the baroclinic pressure gradient terms. Here $\mathcal{F}_i \equiv U/(R_1 R^{-1} GD)^{1/2}$ is the internal Froude number. Both lower and upper limits of integration z and h are nondimensionalized by D.

Since the two hydrostatic pressure gradient terms in (6.17) (the first two terms in the parentheses) need not necessarily cancel each other out, they cannot be larger than the Coriolis acceleration term. It is fair to assume that the second term in parentheses is not greater in order of magnitude than the first since effects of baroclinicity are not generally greater than free-surface tilt. The Coriolis acceleration is then balanced by the pressure gradient due to the free-surface tilt and hence

$$\frac{GH}{U^2} = \frac{FL}{U} = \frac{1}{\epsilon}. \quad (6.19)$$

Also, $P = RGH$ since $P/RU^2 = 1/\epsilon$.

The order of magnitude of the free-surface slope is given by

$$\frac{H}{L} = \frac{1}{\epsilon}\frac{U^2}{GD}\frac{D}{L} = \frac{\delta \mathcal{F}^2}{\epsilon}, \quad (6.20)$$

where $\mathcal{F} \equiv U/(GD)^{1/2}$ is the Froude number. The characteristic free-surface variation over a distance L for $\delta = 10^{-2}$ and different flow parameters is shown in Table 3.

We should comment here that the size of the baroclinic pressure gradient term defined in (6.18) is not correctly characterized by $R_1 GD/L$ since R_1 is the amount by which the density varies from its mean, but the amount by which $\int_z^h R_1 \rho' g dz$, the deviation from mean integrated over the depth, varies over a distance L would be much less than $R_1 GD/L$. Thus, even if $R_1 GD$ is greater than RGH, or $R_1/R > H/D$, it does not mean that the order of magnitude of the baroclinic pressure gradient is greater than that of the Coriolis acceleration.

f. The quasi-nonhydrostatic free-surface model

Using the definition of the baroclinic pressure gradient in (6.18) and dropping the primes from the dimensionless variables, we write the dimensionless equations as

$$\frac{Ds}{Dt} = 0 \quad (6.21a)$$

$$\frac{DT}{Dt} = 0 \quad (6.21b)$$

TABLE 3. Characteristic free-surface variation (U, H) for different flow parameters.

Rossby number ϵ	U (m s^{-1})	H (m)
0.01	0.1	0.1
0.1	1	1
0.1	0.1	0.01

$$\frac{Du}{Dt} + \frac{1}{\epsilon}(gh_x + r_x + \delta q_x - fv + \epsilon\delta bw) = 0 \quad (6.21c)$$

$$\frac{Dv}{Dt} + \frac{1}{\epsilon}(gh_y + r_y + \delta q_y + fu) = 0 \quad (6.21d)$$

$$\frac{Dw}{Dt} + \beta\left(q_z - bu - \epsilon\lambda\frac{(u^2+v^2)}{a}\right) = 0 \quad (6.21e)$$

$$u_x + v_y + \epsilon w_z = 0 \quad (6.21f)$$

$$\rho = \rho(s, T) \quad (6.21g)$$

$$\frac{\partial h}{\partial t} + \frac{\epsilon}{\mathcal{F}^2}\left[\frac{\partial}{\partial x}\left(\int_{-d}^{(H/D)h} u dz\right) + \frac{\partial}{\partial y}\left(\int_{-d}^{(H/D)h} v dz\right)\right] = 0, \quad (6.21h)$$

where

$$\frac{D}{Dt} \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \epsilon w\frac{\partial}{\partial z}.$$

These equations constitute an accurate model for mesoscale flows with a free surface. Forcing terms and any parameterization for subgrid-scale motions may be added to these equations. The boundary conditions for well-posedness are the same as those described in section 5a. However, if pressure alone is prescribed at an outflow boundary, it must consist of both the free-surface elevation and the nonhydrostatic pressure. Since the equations permit a greater deviation from hydrostatic balance than actually exists in nature, it may be appropriate to refer to them as "quasi"-nonhydrostatic. We have discussed the case where two digits of accuracy are maintained in the solution of the nondimensional variables that are $O(1)$. Scaling shows that the characteristic vertical velocity is $\epsilon\delta$ times the horizontal velocity. Hence if $U = 1$ m s^{-1}, $\epsilon = 10^{-1}$, and $\delta = 10^{-2}$, then two digits of accuracy means that the horizontal velocities are accurate to one centimeter per second and vertical velocities are accurate to one-thousandth of this.

7. Conclusions

Building on the approach of Browning et al. (1990), we propose a new system of equations to model mesoscale flows that are associated with currents and eddies in the ocean and are characterized by small Rossby number and aspect ratio. In these equations, we permit a greater deviation from hydrostatic balance than what truly exists to alleviate the stringent accuracy requirement in the vertical momentum equation. The order of magnitude of the solution error in the dimensionless model equations is estimated as equal to the inverse of a modified coefficient in the vertical momentum equation and is maintained small by choosing the coefficient appropriately. These equations model the flow field to a desired accuracy in all three dimensions. Numerical tests performed with different values of the modified coefficient in the following paper, show that the magnitude of the solution error is indeed well within the size predicted by the analysis.

The proposed model is well-posed in domains with open or closed boundaries and does not rely on a minimum size of viscous term. Open boundaries are inevitable in mesoscale modeling since the resolution required for resolving mesoscale eddies is unaffordable for the entire world's oceans.

By achieving both accuracy and well-posedness in open domains, we overcome the two main drawbacks of the traditional hydrostatic (primitive) equations: (i) that they are ill-posed in open domains and therefore require a sufficiently large eddy viscosity to make their solution viable and (ii) that the vertical velocities that are computed from the continuity equation are inaccurate when rotation effects are dominant.

In the nonhydrostatic equations, we find that it is essential to retain the component of the Coriolis acceleration normal to the earth's surface that arises from the tangential component of the Coriolis parameter. This component of the Coriolis acceleration is a dominant term in the vertical momentum equation and is in "geostrophic balance" with the nonhydrostatic pressure gradient in the vertical. Any deviation from this balance induces vertical acceleration in the fluid.

In the horizontal momentum equations, the ratio of the nonhydrostatic to hydrostatic pressure gradients is only of the order of the aspect ratio. Thus, the nonhydrostatic term does not have any significant effect on the horizontal motion in large-scale flows, but retaining it makes the model well-posed and accurate in the vertical.

By formulating the problem with a free surface we are able to provide the correct boundary condition at the free surface and maintain the same relative accuracy in the vertical velocities as in the horizontal. The free surface also enables us to decouple the pressure into hydrostatic and nonhydrostatic components, which we then use to our advantage in the numerical solution procedure.

In the case of mesoscale or other nearly hydrostatic flows, the numerical solution of the nonhydrostatic equations is only marginally more expensive than that of the hydrostatic equations. In many situations, the nonhydrostatic equations have several advantages over the hydrostatic system, and using them is well worth the additional expense.

In a subsequent paper, Mahadevan et al. (1996), we describe the numerical implementation of the model and its application to modeling the mesoscale flow in the Gulf of Mexico. We also explain why the free-surface model formulation makes the numerical solution of the nonhydrostatic equations much less expensive than one would expect.

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APPENDIX

Higher Order Hydrostatic and Incompressibility Approximations

We can now use Eqs. (3.5e) and (3.5f) to examine the accuracy of the hydrostatic and incompressibility approximations. On multiplying (3.5e) by $\epsilon^2\delta^2$, we see that

$$p_z + \rho g - \delta b u - \lambda\epsilon\delta(u^2 + v^2)/a = O(\epsilon^2\delta^2). \quad \text{(A.1)}$$

If we drop the curvature term, then the error in the remaining combination of terms is larger and is given by

$$p_z + \rho g - \delta b u = O(\lambda\epsilon\delta). \quad \text{(A.2)}$$

Neglecting the Coriolis term further increases the error, which is given by

$$p_z + \rho g = O(\delta). \quad \text{(A.3)}$$

Equation (A.3) shows that the hydrostatic approximation $p_z + \rho g = 0$ has an error that is $O(\delta)$. Inclusion of the Coriolis term δbu in (A.2) gives an improved hydrostatic relation $p_z + \rho g - \delta bu = 0$. The error from this new combination of terms is $O(\lambda\epsilon\delta)$; three to four orders less than in the earlier approximation for the range of flow parameters of concern. Inclusion of the term $\lambda\epsilon\delta(u^2 + v^2)/a$ further increases the accuracy, as is shown by (A.1).

Similarly, on multiplying (3.5f) by $\mathcal{M}^2/\epsilon$, we see that

$$(u_x + v_y + \epsilon w_z) + \epsilon \frac{\mathcal{M}^2}{\mathcal{F}_i^2} p_{0z} w - \epsilon \frac{\mathcal{M}^2}{\mathcal{F}^2} g w = O(\mathcal{M}^2/\epsilon). \quad \text{(A.4)}$$

Neglecting the pressure gradient term gives

$$(u_x + v_y + \epsilon w_z) - \epsilon \frac{\mathcal{M}^2}{\mathcal{F}^2} gw = O\left(\epsilon \frac{\mathcal{M}^2}{\mathcal{F}_i^2}\right) \quad \text{(A.5)}$$

and further neglecting the gravity term gives the error in the incompressibility approximation

$$d \equiv u_x + v_y + \epsilon w_z = O\left(\epsilon \frac{\mathcal{M}^2}{\mathcal{F}^2}\right). \quad \text{(A.6)}$$

It is now also evident as to why the hydrostatic system of equations cannot model the vertical velocity accurately for flows with small Rossby number. In the absence of an equation for the vertical velocity,

$$w_z = \epsilon^{-1}(u_x + v_y) \quad \text{(A.7)}$$

is used to compute w. Any error of $O(\epsilon)$ in computing $(u_x + v_y)$ would result in an $O(1)$ error in w.

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